Derivation of a Consistent Flux for the Generalized Wave Continuity Equation (GWCE) in One Dimension

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In this paper, we derive the consistent fluxes for the one-dimensional form of the Generalized Wave Continuity Equation (GWCE). We follow the method proposed by Hughes *et al.* (2000) and Berger and Howington (2002). The consistent fluxes derived in this paper were used in a *Journal of Hydraulic Engineering* paper by Dietrich *et al.* (2006).

Introduction of GWCE

Begin with a simplified form of the one-dimensional GWCE:

$$\frac{\partial^2 \zeta}{\partial t^2} + G\frac{\partial \zeta}{\partial t} - HU\frac{\partial G}{\partial x} - \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(HUU) + gH\frac{\partial \zeta}{\partial x} - E_l\frac{\partial^2}{\partial x^2}(HU) + \tau HU - GHU\right] = 0, \qquad (1)$$

where ζ is the deviation of the free surface elevation from the mean; G is the numerical parameter introduced by Kinnmark (1986); H is the total water surface elevation; U is the velocity in the x-direction; g is the gravitational constant; E_l is the lateral eddy viscosity; and τ is the bottom friction coefficient, here assumed to be a constant and to have units of sec^{-1}. We make the following assumptions:

- Assume the domain of interest for this paper is the two-element domain shown in Figure 1. We are numbering nodes with a subscript i, and we are numbering elements with a subscript j. Each element has a flux defined on its left and right boundaries. For our consistent fluxes to be useful, there must not be a flux discontinuity at any node i. Thus:

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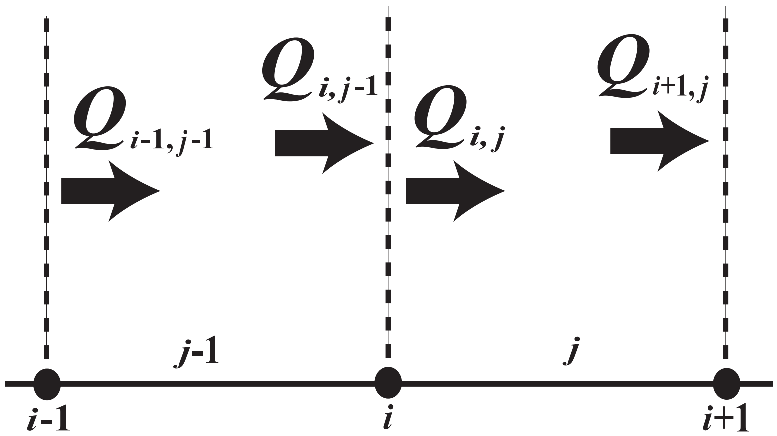

Figure 1. Illustration of the notation used within this report. We are integrating over two elements $j-1$ and j. Each element has a flux defined at its left and right boundaries. For our consistent fluxes to be useful, there must not be a flux discontinuity at node i; that is, $Q_{i,j-1} = Q_{i,j}$.

$$Q_{i,j-1} = Q_{i,j}. \tag{2}$$

To enforce this condition, we will integrate the weight function associated with node i from x_{i-1} to x_{i+1}, use integration by parts to generate a boundary term containing these fluxes, and then solve for them.

- Assume that we are using the standard Galerkin finite element method, so that the basis functions are the same as the weight functions.
- Assume linear Lagrange basis/weight functions:

$$\phi_{i,j}(x) = \frac{x - x_{i+1}}{x_i - x_{i+1}}, \tag{3}$$

and:

$$\phi_{i+1,j}(x) = \frac{x - x_i}{x_{i+1} - x_i}, \tag{4}$$

for element j as in Figure 1. (Note that the basis/weight functions for element $j-1$ are similar to Equations 3 and 4, except both i and j are incremented downward by 1.)

Manipulation of GWCE

To ease the manipulation of the GWCE, we define a temporary variable A to be equal to the stuff in the brackets of Equation 1. Thus we can re-write the equation as:

$$\frac{\partial^2 \zeta}{\partial t^2} + G\frac{\partial \zeta}{\partial t} - HU\frac{\partial G}{\partial x} - \frac{\partial A}{\partial x} = 0 . \tag{5}$$

Multiply Equation 5 by the weight functions associated with node i and integrate over our two-element system to get:

$$\int_{x_{i-1,j-1}}^{x_{i,j-1}} \left(\frac{\partial^2 \zeta}{\partial t^2} + G\frac{\partial \zeta}{\partial t} - HU\frac{\partial G}{\partial x} - \frac{\partial A}{\partial x}\right)\phi_{i,j-1}dx + \int_{x_{i,j}}^{x_{i+1,j}} \left(\frac{\partial^2 \zeta}{\partial t^2} + G\frac{\partial \zeta}{\partial t} - HU\frac{\partial G}{\partial x} - \frac{\partial A}{\partial x}\right)\phi_{i,j}dx = 0 , \tag{6}$$

where we have split the integration over the two elements to emphasize that the weight functions associated with node i are defined in a piece-wise fashion. Now we need to address the A term. Apply integration by parts to those two terms to get:

$$-\int_{x_{i-1,j-1}}^{x_{i,j-1}} \frac{\partial A}{\partial x}\phi_{i,j-1}dx = \int_{x_{i-1,j-1}}^{x_{i,j-1}} A\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}dx - (A\phi_{i,j-1})\Big|_{x_{i-1,j-1}}^{x_{i,j-1}} , \tag{7}$$

and:

$$-\int_{x_{i,j}}^{x_{i+1,j}} \frac{\partial A}{\partial x}\phi_{i,j}dx = \int_{x_{i,j}}^{x_{i+1,j}} A\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}dx - (A\phi_{i,j})\Big|_{x_{i,j}}^{x_{i+1,j}} . \tag{8}$$

To simplify the boundary terms in Equations 7 and 8, we make use of the conservation of mass equation in one dimension:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (HU)}{\partial x} = 0 , \tag{9}$$

and the conservation of momentum equation in one dimension:

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}(g\zeta) + U\frac{\partial U}{\partial x} - E_l\frac{\partial^2 U}{\partial x^2} + \tau U = 0 . \tag{10}$$

We multiply Equation 9 by the x-component of velocity U and Equation 10 by the total water depth H, assume that $\partial\zeta/\partial t = \partial H/\partial t$ because the bathymetry $h = h(x)$, and add the two equations to get:

$$U\frac{\partial H}{\partial t} + U\frac{\partial(HU)}{\partial x} + H\frac{\partial U}{\partial t} + H\frac{\partial}{\partial x}(g\zeta) + HU\frac{\partial U}{\partial x} - HE_l\frac{\partial^2 U}{\partial x^2} + \tau HU = 0\,. \tag{11}$$

Using the product rule, we can combine terms to get:

$$\frac{\partial}{\partial t}(HU) + H\frac{\partial}{\partial x}(g\zeta) + \frac{\partial}{\partial x}(HUU) - HE_l\frac{\partial^2 U}{\partial x^2} + \tau HU = 0\,, \tag{12}$$

and simplify to get:

$$-\frac{\partial}{\partial t}(HU) = gH\frac{\partial \zeta}{\partial x} + \frac{\partial}{\partial x}(HUU) - E_l\frac{\partial^2}{\partial x^2}(HU) + \tau HU\,. \tag{13}$$

Note that all four terms on the right-hand side of Equation 13 appear in the definition of *A*, so we can simplify:

$$A = \frac{\partial}{\partial x}(HUU) + gH\frac{\partial \zeta}{\partial x} - E_l\frac{\partial^2}{\partial x^2}(HU) + \tau HU - GHU \tag{14}$$

$$= -\frac{\partial}{\partial t}(HU) - GHU\,.$$

The last simplification is to assume that $Q = HU$. Then *A* becomes:

$$A = -\left(\frac{\partial Q}{\partial t} + GQ\right), \tag{15}$$

where we have finally introduced the fluxes for which we will be solving. We substitute Equation 15 into the boundary terms in Equations 7 and 8 to get:

$$-\int_{x_{i-1,j-1}}^{x_{i,j-1}} \frac{\partial A}{\partial x}\phi_{i,j-1}dx = \int_{x_{i-1,j-1}}^{x_{i,j-1}} A\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}dx + \left[\left(\frac{\partial Q}{\partial t} + GQ\right)\phi_{i,j-1}\right]\Bigg|_{x_{i-1,j-1}}^{x_{i,j-1}}, \tag{16}$$

and:

$$-\int_{x_{i,j}}^{x_{i+1,j}} \frac{\partial A}{\partial x}\phi_{i,j}dx = \int_{x_{i,j}}^{x_{i+1,j}} A\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}dx + \left[\left(\frac{\partial Q}{\partial t} + GQ\right)\phi_{i,j}\right]\Bigg|_{x_{i,j}}^{x_{i+1,j}}. \tag{17}$$

Then we substitute Equations 16 and 17 into Equation 6 to get:

$$-\left[\left(\frac{\partial Q}{\partial t}+GQ\right)\phi_{i,j-1}\right]\Bigg|_{x_{i-1,j-1}}^{x_{i,j-1}}-\left[\left(\frac{\partial Q}{\partial t}+GQ\right)\phi_{i,j}\right]\Bigg|_{x_{i,j}}^{x_{i+1,j}} \tag{18}$$

$$=\int_{x_{i-1,j-1}}^{x_{i,j-1}}\left(\frac{\partial^2\zeta}{\partial t^2}\phi_{i,j-1}+G\frac{\partial\zeta}{\partial t}\phi_{i,j-1}-HU\frac{\partial G}{\partial x}\phi_{i,j-1}+A\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx$$

$$+\int_{x_{i,j}}^{x_{i+1,j}}\left(\frac{\partial^2\zeta}{\partial t^2}\phi_{i,j}+G\frac{\partial\zeta}{\partial t}\phi_{i,j}-HU\frac{\partial G}{\partial x}\phi_{i,j}+A\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx\,.$$

We can simplify the left-hand side by evaluating the linear Lagrange basis/weight functions:

$$\phi_{i,j-1}(x_{i-1,j-1}) = 0\,, \tag{19}$$

$$\phi_{i,j-1}(x_{i,j-1}) = 1 \tag{20}$$

$$\phi_{i,j}(x_{i+1,j}) = 0\,, \tag{21}$$

and finally:

$$\phi_{i,j}(x_{i,j}) = 1\,. \tag{22}$$

Thus, Equation 18 becomes:

$$-\left(\frac{\partial Q}{\partial t}+GQ\right)\Bigg|_{x_{i,j-1}}+\left(\frac{\partial Q}{\partial t}+GQ\right)\Bigg|_{x_{i,j}} \tag{23}$$

$$=\int_{x_{i-1,j-1}}^{x_{i,j-1}}\left(\frac{\partial^2\zeta}{\partial t^2}\phi_{i,j-1}+G\frac{\partial\zeta}{\partial t}\phi_{i,j-1}-HU\frac{\partial G}{\partial x}\phi_{i,j-1}+A\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx$$

$$+\int_{x_{i,j}}^{x_{i+1,j}}\left(\frac{\partial^2\zeta}{\partial t^2}\phi_{i,j}+G\frac{\partial\zeta}{\partial t}\phi_{i,j}-HU\frac{\partial G}{\partial x}\phi_{i,j}+A\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx\,.$$

Note that we have not cancelled the remaining boundary terms on the left-hand side of Equation 23, even though they have opposite signs and are evaluated at the same spatial location (because $x_{i,j-1} = x_{i,j} = x_i$, from Figure 1). We keep these terms and solve for the consistent fluxes within them. We can break apart Equation 23 into equations for the elements on either side of x_i:

$$\left(\frac{\partial Q}{\partial t} + GQ\right)\bigg|_{x_{i,j-1}} = -\int_{x_{i-1,j-1}}^{x_{i,j-1}} \left(\frac{\partial^2 \zeta}{\partial t^2}\phi_{i,j-1} + G\frac{\partial \zeta}{\partial t}\phi_{i,j-1} - HU\frac{\partial G}{\partial x}\phi_{i,j-1} + A\frac{d\phi_{i,j-1}}{dx}\right)dx \tag{24}$$

$$= \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\frac{\partial^2 \zeta}{\partial t^2}\phi_{i,j-1} + G\frac{\partial \zeta}{\partial t}\phi_{i,j-1} - HU\frac{\partial G}{\partial x}\phi_{i,j-1} + A\frac{d\phi_{i,j-1}}{dx}\right)dx\,,$$

and:

$$\left(\frac{\partial Q}{\partial t} + GQ\right)\bigg|_{x_{i,j}} = \int_{x_{i,j}}^{x_{i+1,j}} \left(\frac{\partial^2 \zeta}{\partial t^2}\phi_{i,j} + G\frac{\partial \zeta}{\partial t}\phi_{i,j} - HU\frac{\partial G}{\partial x}\phi_{i,j} + A\frac{d\phi_{i,j}}{dx}\right)dx\,. \tag{25}$$

The quantity $\partial Q/\partial t + GQ$ derived from Equation 24 should be the same as the quantity $\partial Q/\partial t + GQ$ derived from Equation 25. Let's try to prove it.

For the element to the left of the node:

We begin with Equation 24, simplify by realizing that $\partial G/\partial x = 0$ for our test cases, and substitute for the dummy variable A:

$$\left(\frac{\partial Q}{\partial t} + GQ\right)\bigg|_{x_{i,j-1}} = \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\frac{\partial^2 \zeta}{\partial t^2}\phi_{i,j-1} + G\frac{\partial \zeta}{\partial t}\phi_{i,j-1} + \frac{\partial}{\partial x}(HUU)\frac{d\phi_{i,j-1}}{dx} + gH\frac{\partial \zeta}{\partial x}\frac{d\phi_{i,j-1}}{dx}\right. \tag{26}$$
$$\left. - E_l\frac{\partial^2}{\partial x^2}(HU)\frac{d\phi_{i,j-1}}{dx} + \tau HU\frac{d\phi_{i,j-1}}{dx} - GHU\frac{d\phi_{i,j-1}}{dx}\right)dx\,.$$

Remember that the subscripts i and j refer to nodes and elements, respectively. There are no implied sums in Equation 26. We can solve for the flux Q by recognizing that Equation 26 has the form:

$$\frac{\partial u}{\partial t} - au = f(t)\,, \tag{27}$$

which has the following analytical solution:

$$u = \int_0^t e^{a(t-s)}f(s)ds + e^{at}u_0\,, \tag{28}$$

which can be applied to our situation as:

$$Q^{t+1} = e^{-G\Delta t}\int_0^{\Delta t} e^{Gs} f(s)ds + e^{-G\Delta t}Q^t. \qquad (29)$$

Now all we need is a discrete form of $f(s)$, which represents the right-hand side of Equation 26. Let's go term-by term in that right-hand side.

First term of left element:

For the first term:

$$\int_{x_{i,j-1}}^{x_{i-1,j-1}} \frac{\partial^2 \zeta}{\partial t^2} \phi_{i,j-1} dx, \qquad (30)$$

we approximate the deviation of the free surface elevation from the mean by using the method of separation of variables:

$$\zeta(x,t) = \zeta_{i-1,j-1}(t)\phi_{i-1,j-1}(x) + \zeta_{i,j-1}(t)\phi_{i,j-1}(x), \qquad (31)$$

so that the first term becomes:

$$\int_{x_{i,j-1}}^{x_{i-1,j-1}} \frac{\partial^2 \zeta}{\partial t^2} \phi_{i,j-1} dx = \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\frac{\partial^2 \zeta_{i-1,j-1}}{\partial t^2}(\phi_{i-1,j-1}\phi_{i,j-1}) + \frac{\partial^2 \zeta_{i,j-1}}{\partial t^2}(\phi_{i,j-1}\phi_{i,j-1}) \right) dx \qquad (32)$$

$$= \frac{\partial^2 \zeta_{i-1,j-1}}{\partial t^2} \int_{x_{i,j-1}}^{x_{i-1,j-1}} (\phi_{i-1,j-1}\phi_{i,j-1}) dx + \frac{\partial^2 \zeta_{i,j-1}}{\partial t^2} \int_{x_{i,j-1}}^{x_{i-1,j-1}} (\phi_{i,j-1}\phi_{i,j-1}) dx$$

$$= \frac{\partial^2 \zeta_{i-1,j-1}}{\partial t^2}\left(\frac{-\Delta x}{6}\right) + \frac{\partial^2 \zeta_{i,j-1}}{\partial t^2}\left(\frac{-\Delta x}{3}\right).$$

Second term of left element:

For the second term:

$$\int_{x_{i,j-1}}^{x_{i-1,j-1}} G \frac{\partial \zeta}{\partial t} \phi_{i,j-1} dx, \qquad (33)$$

we use the same separation of variables expression as in Equation 31, and the expression becomes:

$$\int_{x_{i,j-1}}^{x_{i-1,j-1}} G\frac{\partial\zeta}{\partial t}\phi_{i,j-1}dx \tag{34}$$

$$= G\frac{\partial\zeta_{i-1,j-1}}{\partial t}\int_{x_{i,j-1}}^{x_{i-1,j-1}}(\phi_{i-1,j-1}\phi_{i,j-1})dx + G\frac{\partial\zeta_{i,j-1}}{\partial t}\int_{x_{i,j-1}}^{x_{i-1,j-1}}(\phi_{i,j-1}\phi_{i,j-1})dx$$

$$= G\frac{\partial\zeta_{i-1,j-1}}{\partial t}\left(\frac{-\Delta x}{6}\right) + G\frac{\partial\zeta_{i,j-1}}{\partial t}\left(\frac{-\Delta x}{3}\right).$$

Third term of left element:

For the third term:

$$\int_{x_{i,j-1}}^{x_{i-1,j-1}} \frac{\partial}{\partial x}(HUU)\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}dx, \tag{35}$$

we expand the first term:

$$\int_{x_{i,j-1}}^{x_{i-1,j-1}} \frac{\partial}{\partial x}(HUU)\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}dx = \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left[\left(H\frac{\partial}{\partial x}(UU) + UU\frac{\partial H}{\partial x}\right)\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right]dx \tag{36}$$

$$= \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left[\left(HU\frac{\partial U}{\partial x} + HU\frac{\partial U}{\partial x} + UU\frac{\partial H}{\partial x}\right)\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right]dx$$

$$= \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left[\left(2HU\frac{\partial U}{\partial x} + UU\frac{\partial H}{\partial x}\right)\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right]dx.$$

Let’s use separation of variables to simplify:

$$H = H_{i-1,j-1}(t)\phi_{i-1,j-1}(x) + H_{i,j-1}(t)\phi_{i,j-1}(x), \tag{37}$$

and:

$$U = U_{i-1,j-1}(t)\phi_{i-1,j-1}(x) + U_{i,j-1}(t)\phi_{i,j-1}(x). \tag{38}$$

Thus:

$$\int_{x_{i,j-1}}^{x_{i-1,j-1}} \frac{\partial}{\partial x}(HUU)\frac{\partial \phi_{i,j-1}}{\partial x}dx = \int_{x_{i,j-1}}^{x_{i-1,j-1}} [2(H_{i-1,j-1}\phi_{i-1,j-1} + H_{i,j-1}\phi_{i,j-1}) \tag{39}$$

$$(U_{i-1,j-1}\phi_{i-1,j-1} + U_{i,j-1}\phi_{i,j-1})\frac{\partial}{\partial x}(U_{i-1,j-1}\phi_{i-1,j-1} + U_{i,j-1}\phi_{i,j-1})\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}$$

$$+ (U_{i-1,j-1}\phi_{i-1,j-1} + U_{i,j-1}\phi_{i,j-1})(U_{i-1,j-1}\phi_{i-1,j-1} + U_{i,j-1}\phi_{i,j-1})$$

$$\frac{\partial}{\partial x}(H_{i-1,j-1}\phi_{i-1,j-1} + H_{i,j-1}\phi_{i,j-1})\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\Big]dx$$

$$= \int_{x_{i,j-1}}^{x_{i-1,j-1}} \Big[\Big(2H_{i-1,j-1}\phi_{i-1,j-1}U_{i-1,j-1}\phi_{i-1,j-1}U_{i-1,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}$$

$$+ 2H_{i-1,j-1}\phi_{i-1,j-1}U_{i-1,j-1}\phi_{i-1,j-1}U_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}$$

$$+ 2H_{i-1,j-1}\phi_{i-1,j-1}U_{i,j-1}\phi_{i,j-1}U_{i-1,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}$$

$$+ 2H_{i-1,j-1}\phi_{i-1,j-1}U_{i,j-1}\phi_{i,j-1}U_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}$$

$$+ 2H_{i,j-1}\phi_{i,j-1}U_{i-1,j-1}\phi_{i-1,j-1}U_{i-1,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}$$

$$+ 2H_{i,j-1}\phi_{i,j-1}U_{i-1,j-1}\phi_{i-1,j-1}U_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}$$

$$+ 2H_{i,j-1}\phi_{i,j-1}U_{i,j-1}\phi_{i,j-1}U_{i-1,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}$$

$$+ 2H_{i,j-1}\phi_{i,j-1}U_{i,j-1}\phi_{i,j-1}U_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\Big)$$

$$+ \Big(U_{i-1,j-1}\phi_{i-1,j-1}U_{i-1,j-1}\phi_{i-1,j-1}H_{i-1,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}$$

$$+ U_{i-1,j-1}\phi_{i-1,j-1}U_{i-1,j-1}\phi_{i-1,j-1}H_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}$$

$$+ U_{i-1,j-1}\phi_{i-1,j-1}U_{i,j-1}\phi_{i,j-1}H_{i-1,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}$$

$$+ U_{i-1,j-1}\phi_{i-1,j-1}U_{i,j-1}\phi_{i,j-1}H_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}$$

$$+ U_{i,j-1}\phi_{i,j-1}U_{i-1,j-1}\phi_{i-1,j-1}H_{i-1,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}$$

$$+ U_{i,j-1}\phi_{i,j-1}U_{i-1,j-1}\phi_{i-1,j-1}H_{i,j-1}\frac{d\phi_{i,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}$$

$$+ U_{i,j-1}\phi_{i,j-1}U_{i,j-1}\phi_{i,j-1}H_{i-1,j-1}\frac{d\phi_{i-1,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}$$

$$+ U_{i,j-1}\phi_{i,j-1}U_{i,j-1}\phi_{i,j-1}H_{i,j-1}\frac{d\phi_{i,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}\Bigg)\Bigg]dx$$

$$= 2H_{i-1,j-1}U^2_{i-1,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i-1,j-1}\phi_{i-1,j-1}\frac{d\phi_{i-1,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}\right)dx$$

$$+ 2H_{i-1,j-1}U_{i-1,j-1}U_{i,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i-1,j-1}\phi_{i-1,j-1}\frac{d\phi_{i,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}\right)dx$$

$$+ 2H_{i-1,j-1}U_{i,j-1}U_{i-1,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i-1,j-1}\phi_{i,j-1}\frac{d\phi_{i-1,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}\right)dx$$

$$+ 2H_{i-1,j-1}U_{i,j-1}U_{i,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i-1,j-1}\phi_{i,j-1}\frac{d\phi_{i,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}\right)dx$$

$$+ 2H_{i,j-1}U_{i-1,j-1}U_{i-1,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i,j-1}\phi_{i-1,j-1}\frac{d\phi_{i-1,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}\right)dx$$

$$+ 2H_{i,j-1}U_{i-1,j-1}U_{i,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i,j-1}\phi_{i-1,j-1}\frac{d\phi_{i,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}\right)dx$$

$$+ 2H_{i,j-1}U_{i,j-1}U_{i-1,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i,j-1}\phi_{i,j-1}\frac{d\phi_{i-1,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}\right)dx$$

$$+ 2H_{i,j-1}U_{i,j-1}U_{i,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i,j-1}\phi_{i,j-1}\frac{d\phi_{i,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}\right)dx$$

$$+ U_{i-1,j-1}U_{i-1,j-1}H_{i-1,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i-1,j-1}\phi_{i-1,j-1}\frac{d\phi_{i-1,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}\right)dx$$

$$+ U_{i-1,j-1}U_{i-1,j-1}H_{i,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i-1,j-1}\phi_{i-1,j-1}\frac{d\phi_{i,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}\right)dx$$

$$+ U_{i-1,j-1}U_{i,j-1}H_{i-1,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i-1,j-1}\phi_{i,j-1}\frac{d\phi_{i-1,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}\right)dx$$

$$+ U_{i-1,j-1}U_{i,j-1}H_{i,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i-1,j-1}\phi_{i,j-1}\frac{d\phi_{i,j-1}}{dx}\frac{d\phi_{i,j-1}}{dx}\right)dx$$

$$+ U_{i,j-1}U_{i-1,j-1}H_{i-1,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i,j-1}\phi_{i-1,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx$$

$$+ U_{i,j-1}U_{i-1,j-1}H_{i,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i,j-1}\phi_{i-1,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx$$

$$+ U_{i,j-1}U_{i,j-1}H_{i-1,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i,j-1}\phi_{i,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx$$

$$+ U_{i,j-1}U_{i,j-1}H_{i,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i,j-1}\phi_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx$$

$$= 2H_{i-1,j-1}U_{i-1,j-1}^2\left(\frac{1}{3\Delta x}\right) + 2H_{i-1,j-1}U_{i-1,j-1}U_{i,j-1}\left(\frac{-1}{3\Delta x}\right)$$

$$+ 2H_{i-1,j-1}U_{i,j-1}U_{i-1,j-1}\left(\frac{1}{6\Delta x}\right) + 2H_{i-1,j-1}U_{i,j-1}^2\left(\frac{-1}{6\Delta x}\right)$$

$$+ 2H_{i,j-1}U_{i-1,j-1}^2\left(\frac{1}{6\Delta x}\right) + 2H_{i,j-1}U_{i-1,j-1}U_{i,j-1}\left(\frac{-1}{6\Delta x}\right)$$

$$+ 2H_{i,j-1}U_{i,j-1}U_{i-1,j-1}\left(\frac{1}{3\Delta x}\right) + 2H_{i,j-1}U_{i,j-1}^2\left(\frac{-1}{3\Delta x}\right)$$

$$+ U_{i-1,j-1}^2H_{i-1,j-1}\left(\frac{1}{3\Delta x}\right) + U_{i-1,j-1}^2H_{i,j-1}\left(\frac{-1}{3\Delta x}\right)$$

$$+ U_{i-1,j-1}U_{i,j-1}H_{i-1,j-1}\left(\frac{1}{6\Delta x}\right) + U_{i-1,j-1}U_{i,j-1}H_{i,j-1}\left(\frac{-1}{6\Delta x}\right)$$

$$+ U_{i,j-1}U_{i-1,j-1}H_{i-1,j-1}\left(\frac{1}{6\Delta x}\right) + U_{i,j-1}U_{i-1,j-1}H_{i,j-1}\left(\frac{-1}{6\Delta x}\right)$$

$$+ U_{i,j-1}^2H_{i-1,j-1}\left(\frac{1}{3\Delta x}\right) + U_{i,j-1}^2H_{i,j-1}\left(\frac{-1}{3\Delta x}\right)$$

$$= \frac{1}{\Delta x}[H_{i-1,j-1}U_{i-1,j-1}^2 - H_{i,j-1}U_{i,j-1}^2],$$

because all of the other product terms can be cancelled.

Fourth term of left element:

For the fourth term:

$$\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(gH\frac{\partial\zeta\partial\phi_{i,j-1}}{\partial x\partial x}\right)dx = \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(g(h+\zeta)\frac{\partial\zeta\partial\phi_{i,j-1}}{\partial x\partial x}\right)dx \tag{40}$$

$$= \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(gh\frac{\partial\zeta\partial\phi_{i,j-1}}{\partial x\partial x}\right)dx + \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(g\zeta\frac{\partial\zeta\partial\phi_{i,j-1}}{\partial x\partial x}\right)dx .$$

Approximate h and ζ by using separation of variables:

$$h = h_{i-1,j-1}(t)\phi_{i-1,j-1}(x) + h_{i,j-1}(t)\phi_{i,j-1}(x) , \tag{41}$$

and:

$$\zeta = \zeta_{i-1,j-1}(t)\phi_{i-1,j-1}(x) + \zeta_{i,j-1}(t)\phi_{i,j-1}(x) . \tag{42}$$

Expanding Equation 40 gives:

$$\begin{aligned}
&\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(g(h_{i-1,j-1}\phi_{i-1,j-1} + h_{i,j-1}\phi_{i,j-1})\frac{\partial}{\partial x}(\zeta_{i-1,j-1}\phi_{i-1,j-1})\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx \\
&+ \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(g(h_{i-1,j-1}\phi_{i-1,j-1} + h_{i,j-1}\phi_{i,j-1})\frac{\partial}{\partial x}(\zeta_{i,j-1}\phi_{i,j-1})\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx \\
&+ \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(g(\zeta_{i-1,j-1}\phi_{i-1,j-1} + \zeta_{i,j-1}\phi_{i,j-1})\frac{\partial}{\partial x}(\zeta_{i-1,j-1}\phi_{i-1,j-1})\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx \\
&+ \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(g(\zeta_{i-1,j-1}\phi_{i-1,j-1} + \zeta_{i,j-1}\phi_{i,j-1})\frac{\partial}{\partial x}(\zeta_{i,j-1}\phi_{i,j-1})\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx
\end{aligned} \tag{43}$$

$$\begin{aligned}
&= \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(gh_{i-1,j-1}\phi_{i-1,j-1}\zeta_{i-1,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx \\
&+ \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(gh_{i,j-1}\phi_{i,j-1}\zeta_{i-1,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx \\
&+ \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(gh_{i-1,j-1}\phi_{i-1,j-1}\zeta_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx \\
&+ \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(gh_{i,j-1}\phi_{i,j-1}\zeta_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx \\
&+ \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(g\zeta_{i-1,j-1}\phi_{i-1,j-1}\zeta_{i-1,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx
\end{aligned}$$

$$+ \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(g\zeta_{i,j-1}\phi_{i,j-1}\zeta_{i-1,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$+ \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(g\zeta_{i-1,j-1}\phi_{i-1,j-1}\zeta_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$+ \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(g\zeta_{i,j-1}\phi_{i,j-1}\zeta_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$= gh_{i-1,j-1}\zeta_{i-1,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\phi_{i-1,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$+ gh_{i,j-1}\zeta_{i-1,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\phi_{i,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$+ gh_{i-1,j-1}\zeta_{i,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\phi_{i-1,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$+ gh_{i,j-1}\zeta_{i,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\phi_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$+ g\zeta_{i-1,j-1}^2\int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\phi_{i-1,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$+ g\zeta_{i,j-1}\zeta_{i-1,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\phi_{i,j-1}\frac{\mathrm{d}\phi_{i-1,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$+ g\zeta_{i-1,j-1}\zeta_{i,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\phi_{i-1,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$+ g\zeta_{i,j-1}^2\int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\phi_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$= gh_{i-1,j-1}\zeta_{i-1,j-1}\left(\frac{1}{2\Delta x}\right) + gh_{i,j-1}\zeta_{i-1,j-1}\left(\frac{1}{2\Delta x}\right)$$

$$+ gh_{i-1,j-1}\zeta_{i,j-1}\left(\frac{-1}{2\Delta x}\right) + gh_{i,j-1}\zeta_{i,j-1}\left(\frac{-1}{2\Delta x}\right)$$

$$+ g\zeta_{i-1,j-1}^2\left(\frac{1}{2\Delta x}\right) + g\zeta_{i,j-1}\zeta_{i-1,j-1}\left(\frac{1}{2\Delta x}\right)$$

$$+ g\zeta_{i-1,j-1}\zeta_{i,j-1}\left(\frac{-1}{2\Delta x}\right) + g\zeta_{i,j-1}^2\left(\frac{-1}{2\Delta x}\right)$$

$$= \frac{g}{2\Delta x}(\zeta_{i-1,j-1} - \zeta_{i,j-1})[(h_{i-1,j-1} + h_{i,j-1}) + (\zeta_{i-1,j-1} + \zeta_{i,j-1})]$$

$$= \frac{g}{2\Delta x}(\zeta_{i-1,j-1} - \zeta_{i,j-1})(H_{i-1,j-1} + H_{i,j-1}) .$$

Fifth term of left element:

For the fifth term:

$$-\int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(E_l \frac{\partial^2}{\partial x^2}(HU) \frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx , \tag{44}$$

which can be simplified by substituting the conservation of mass equation:

$$-\int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(E_l \frac{\partial^2}{\partial x^2}(HU) \frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx = -\int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(E_l \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(HU) \right) \frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx \tag{45}$$

$$= -\int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(E_l \frac{\partial}{\partial x}\left(-\frac{\partial H}{\partial t} \right) \frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$= \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(E_l \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}(h + \zeta) \right) \frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$= \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(E_l \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial t} + \frac{\partial \zeta}{\partial t} \right) \frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$= \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(E_l \frac{\partial^2 \zeta}{\partial t \partial x} \frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx$$

$$= E_l \frac{\partial}{\partial t} \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\frac{\partial \zeta}{\partial x} \frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x} \right) dx ,$$

which can be approximated as before:

$$= E_l \frac{\partial}{\partial t} \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\frac{\partial}{\partial x} (\zeta_{i-1,j-1} \phi_{i-1,j-1} + \zeta_{i,j-1} \phi_{i,j-1}) \frac{d\phi_{i,j-1}}{dx} \right) dx \tag{46}$$

$$= E_l \frac{\partial}{\partial t} \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\left(\zeta_{i-1,j-1} \frac{d\phi_{i-1,j-1}}{dx} + \zeta_{i,j-1} \frac{d\phi_{i,j-1}}{dx} \right) \frac{d\phi_{i,j-1}}{dx} \right) dx$$

$$= E_l \frac{d\zeta_{i-1,j-1}}{dt} \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\frac{d\phi_{i-1,j-1}}{dx} \frac{d\phi_{i,j-1}}{dx} \right) dx + E_l \frac{d\zeta_{i,j-1}}{dt} \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\frac{d\phi_{i,j-1}}{dx} \frac{d\phi_{i,j-1}}{dx} \right) dx$$

$$= E_l \frac{d\zeta_{i-1,j-1}}{dt} \left(\frac{1}{\Delta x} \right) + E_l \frac{d\zeta_{i,j-1}}{dt} \left(\frac{-1}{\Delta x} \right)$$

$$= \frac{E_l}{\Delta x} \left(\frac{d\zeta_{i-1,j-1}}{dt} - \frac{d\zeta_{i,j-1}}{dt} \right).$$

Sixth term of left element:

For the sixth term:

$$\int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\tau HU \frac{d\phi_{i,j-1}}{dx} \right) dx, \tag{47}$$

we approximate the product *HU* with the following interpolation:

$$HU = (HU)_{i-1,j-1}(t)\phi_{i-1,j-1}(x) + (HU)_{i,j-1}(t)\phi_{i,j-1}(x), \tag{48}$$

so that the sixth term becomes:

$$\int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\tau HU \frac{d\phi_{i,j-1}}{dx} \right) dx = \int_{x_{i,j-1}}^{x_{i-1,j-1}} [(\tau_{i-1,j-1}\phi_{i-1,j-1} \tag{49}$$

$$+ \tau_{i,j-1}\phi_{i,j-1})((HU)_{i-1,j-1}\phi_{i-1,j-1} + (HU)_{i,j-1}\phi_{i,j-1}) \frac{d\phi_{i,j-1}}{dx} \Bigg] dx$$

$$= \int_{x_{i,j-1}}^{x_{i-1,j-1}} \left(\tau_{i-1,j-1}\phi_{i-1,j-1}(HU)_{i-1,j-1}\phi_{i-1,j-1} \frac{d\phi_{i,j-1}}{dx} \right) dx$$

$$+\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\tau_{i-1,j-1}\phi_{i-1,j-1}(HU)_{i,j-1}\phi_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx$$

$$+\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\tau_{i,j-1}\phi_{i,j-1}(HU)_{i-1,j-1}\phi_{i-1,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx$$

$$+\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\tau_{i,j-1}\phi_{i,j-1}(HU)_{i,j-1}\phi_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx$$

$$= \tau_{i-1,j-1}(HU)_{i-1,j-1}\left(\frac{-1}{3}\right) + \tau_{i-1,j-1}(HU)_{i,j-1}\left(\frac{-1}{6}\right)$$

$$+ \tau_{i,j-1}(HU)_{i-1,j-1}\left(\frac{-1}{6}\right) + \tau_{i,j-1}(HU)_{i,j-1}\left(\frac{-1}{3}\right)$$

$$= (HU)_{i-1,j-1}\left(\frac{-\tau_{i-1,j-1}}{3} + \frac{-\tau_{i,j-1}}{6}\right) + (HU)_{i,j-1}\left(\frac{-\tau_{i-1,j-1}}{6} + \frac{-\tau_{i,j-1}}{3}\right).$$

Seventh term of left element:

For the seventh term:

$$-\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(GHU\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx, \quad (50)$$

we make the same approximations as before:

$$= -\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(G((HU)_{i-1,j-1}\phi_{i-1,j-1} + (HU)_{i,j-1}\phi_{i,j-1})\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx \quad (51)$$

$$= -\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(G(HU)_{i-1,j-1}\phi_{i-1,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx - \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(G(HU)_{i,j-1}\phi_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx$$

$$= -G(HU)_{i-1,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i-1,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx - G(HU)_{i,j-1}\int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\phi_{i,j-1}\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx$$

$$= -G(HU)_{i-1,j-1}\left(\frac{-1}{2}\right) - G(HU)_{i,j-1}\left(\frac{-1}{2}\right)$$

$$= (HU)_{i-1,j-1}\left(\frac{G}{2}\right) + (HU)_{i,j-1}\left(\frac{G}{2}\right).$$

Pulling it all together:

Then Equation 26 can be rewritten with a discretized right-hand side:

$$\left(\frac{\partial Q}{\partial t} + GQ\right)\bigg|_{x_{i,j-1}} = \left[\frac{\partial^2 \zeta_{i-1,j-1}}{\partial t^2}\left(\frac{-\Delta x}{6}\right) + \frac{\partial^2 \zeta_{i,j-1}}{\partial t^2}\left(\frac{-\Delta x}{3}\right)\right] \tag{52}$$

$$+ \left[G\frac{\partial \zeta_{i-1,j-1}}{\partial t}\left(\frac{-\Delta x}{6}\right) + G\frac{\partial \zeta_{i,j-1}}{\partial t}\left(\frac{-\Delta x}{3}\right)\right] + \left[\frac{1}{\Delta x}[H_{i-1,j-1}U^2_{i-1,j-1} - H_{i,j-1}U^2_{i,j-1}]\right]$$

$$+ \left[\frac{g}{2\Delta x}(\zeta_{i-1,j-1} - \zeta_{i,j-1})(H_{i-1,j-1} + H_{i,j-1})\right] + \left[\frac{E_l}{\Delta x}\left(\frac{\mathrm{d}\zeta_{i-1,j-1}}{\mathrm{d}t} - \frac{\mathrm{d}\zeta_{i,j-1}}{\mathrm{d}t}\right)\right]$$

$$+ (HU)_{i-1,j-1}\left(\frac{-\tau_{i-1,j-1}}{3} + \frac{-\tau_{i,j-1}}{6}\right) + (HU)_{i,j-1}\left(\frac{-\tau_{i-1,j-1}}{6} + \frac{-\tau_{i,j-1}}{3}\right)$$

$$+ (HU)_{i-1,j-1}\left(\frac{G}{2}\right) + (HU)_{i,j-1}\left(\frac{G}{2}\right),$$

which can be simplified slightly to become:

$$\left(\frac{\partial Q}{\partial t} + GQ\right)\bigg|_{x_{i,j-1}} = \left[\frac{\partial^2 \zeta_{i-1,j-1}}{\partial t^2}\left(\frac{-\Delta x}{6}\right) + \frac{\partial^2 \zeta_{i,j-1}}{\partial t^2}\left(\frac{-\Delta x}{3}\right)\right] \tag{53}$$

$$+ \left[\frac{\partial \zeta_{i-1,j-1}}{\partial t}\left(\frac{E_l}{\Delta x} - \frac{G\Delta x}{6}\right) - \frac{\partial \zeta_{i,j-1}}{\partial t}\left(\frac{E_l}{\Delta x} + \frac{G\Delta x}{3}\right)\right]$$

$$+ H_{i-1,j-1}\left[\frac{U^2_{i-1,j-1}}{\Delta x} + \frac{g}{2\Delta x}(\zeta_{i-1,j-1} - \zeta_{i,j-1})\right] + H_{i,j-1}\left[\frac{-U^2_{i-1,j-1}}{\Delta x} + \frac{g}{2\Delta x}(\zeta_{i-1,j-1} - \zeta_{i,j-1})\right]$$

$$+ (HU)_{i-1,j-1}\left[\frac{-\tau_{i-1,j-1}}{3} + \frac{-\tau_{i,j-1}}{6} + \frac{G}{2}\right] + (HU)_{i,j-1}\left[\frac{-\tau_{i-1,j-1}}{6} + \frac{-\tau_{i,j-1}}{3} + \frac{G}{2}\right].$$

This gives us a differential equation in time that we can solve using Equation 29. Before we solve, let's generate a similar equation for the right element.

For the element to the right of the node:

The element to the right of the node is governed by Equation 25. We substitute the long form of A and assume G does not vary spatially to get:

$$\left.\left(\frac{\partial Q}{\partial t}+GQ\right)\right|_{x_{i,j}} = \int_{x_{i,j}}^{x_{i+1,j}}\left(\frac{\partial^2\zeta}{\partial t^2}\phi_{i,j}+G\frac{\partial\zeta}{\partial t}\phi_{i,j}\right. \tag{54}$$

$$\left.+\frac{\partial}{\partial x}(HUU)\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}+gH\frac{\partial\zeta}{\partial x}\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}-E_l\frac{\partial^2}{\partial x^2}(HU)\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}+\tau HU\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}-GHU\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx\,.$$

Again, we need a discretized form of the right-hand side. There are seven terms, so let's handle each term individually.

First term of right element:

For the first term:

$$\int_{x_{i,j}}^{x_{i+1,j}}\left(\frac{\partial^2\zeta}{\partial t^2}\phi_{i,j}\right)dx = \int_{x_{i,j}}^{x_{i+1,j}}\left(\frac{\partial^2}{\partial t^2}(\zeta_{i,j}\phi_{i,j}+\zeta_{i+1,j}\phi_{i+1,j})\phi_{i,j}\right)dx \tag{55}$$

$$= \frac{\mathrm{d}^2\zeta_{i,j}}{\mathrm{d}t^2}\int_{x_{i,j}}^{x_{i+1,j}}(\phi_{i,j}\phi_{i,j})dx+\frac{\mathrm{d}^2\zeta_{i+1,j}}{\mathrm{d}t^2}\int_{x_{i,j}}^{x_{i+1,j}}(\phi_{i+1,j}\phi_{i,j})dx$$

$$= \frac{\mathrm{d}^2\zeta_{i,j}}{\mathrm{d}t^2}\left(\frac{\Delta x}{3}\right)+\frac{\mathrm{d}^2\zeta_{i+1,j}}{\mathrm{d}t^2}\left(\frac{\Delta x}{6}\right).$$

Second term of right element:

For the second term:

$$\int_{x_{i,j}}^{x_{i+1,j}}\left(G\frac{\partial\zeta}{\partial t}\phi_{i,j}\right)dx = \int_{x_{i,j}}^{x_{i+1,j}}\left(G\frac{\partial}{\partial t}(\zeta_{i,j}\phi_{i,j}+\zeta_{i+1,j}\phi_{i+1,j})\phi_{i,j}\right)dx \tag{56}$$

$$= G\frac{\mathrm{d}\zeta_{i,j}}{\mathrm{d}t}\int_{x_{i,j}}^{x_{i+1,j}}(\phi_{i,j}\phi_{i,j})dx+G\frac{\mathrm{d}\zeta_{i+1,j}}{\mathrm{d}t}\int_{x_{i,j}}^{x_{i+1,j}}(\phi_{i+1,j}\phi_{i,j})dx$$

$$= G\frac{\mathrm{d}\zeta_{i,j}}{\mathrm{d}t}\left(\frac{\Delta x}{3}\right) + G\frac{\mathrm{d}\zeta_{i+1,j}}{\mathrm{d}t}\left(\frac{\Delta x}{6}\right).$$

Third term of right element:

For the third term:

$$\int_{x_{i,j}}^{x_{i+1,j}}\left(\frac{\partial}{\partial x}(HUU)\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx, \tag{57}$$

we can approximate the H and U terms individually, as we did above. However, you get the same answer (and the math is much simpler) if you approximate the product as:

$$HUU = (HUU)_{i,j}\phi_{i,j} + (HUU)_{i+1,j}\phi_{i+1,j}. \tag{58}$$

Then Equation 57 becomes:

$$\int_{x_{i,j}}^{x_{i+1,j}}\left(\frac{\partial}{\partial x}(HUU)\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx = \int_{x_{i,j}}^{x_{i+1,j}}\left(\frac{\partial}{\partial x}[(HUU)_{i,j}\phi_{i,j} + (HUU)_{i+1,j}\phi_{i+1,j}]\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx \tag{59}$$

$$= (HUU)_{i,j}\int_{x_{i,j}}^{x_{i+1,j}}\left(\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx + (HUU)_{i+1,j}\int_{x_{i,j}}^{x_{i+1,j}}\left(\frac{\mathrm{d}\phi_{i+1,j}}{\mathrm{d}x}\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx$$

$$= (HUU)_{i,j}\left(\frac{1}{\Delta x}\right) + (HUU)_{i+1,j}\left(\frac{-1}{\Delta x}\right)$$

$$= \frac{1}{\Delta x}[H_{i,j}U_{i,j}^2 - H_{i+1,j}U_{i+1,j}^2],$$

which is similar to what we got for this term integrated over the left element, but we achieved it in much less work.

Fourth term of right element:

For the fourth term:

$$\int_{x_{i,j}}^{x_{i+1,j}}\left(gH\frac{\partial\zeta}{\partial x}\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx = \int_{x_{i,j}}^{x_{i+1,j}}\left(g(H_{i,j}\phi_{i,j} + H_{i+1,j}\phi_{i+1,j})\frac{\partial}{\partial x}(\zeta_{i,j}\phi_{i,j} + \zeta_{i+1,j}\phi_{i+1,j})\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx \tag{60}$$

$$= \int_{x_{i,j}}^{x_{i+1,j}} \left(g(H_{i,j}\phi_{i,j} + H_{i+1,j}\phi_{i+1,j}) \left(\zeta_{i,j}\frac{d\phi_{i,j}}{dx} + \zeta_{i+1,j}\frac{d\phi_{i+1,j}}{dx} \right) \frac{d\phi_{i,j}}{dx} \right) dx$$

$$= gH_{i,j}\zeta_{i,j}\int_{x_{i,j}}^{x_{i+1,j}} \left(\phi_{i,j}\frac{d\phi_{i,j}}{dx}\frac{d\phi_{i,j}}{dx} \right) dx + gH_{i,j}\zeta_{i+1,j}\int_{x_{i,j}}^{x_{i+1,j}} \left(\phi_{i,j}\frac{d\phi_{i+1,j}}{dx}\frac{d\phi_{i,j}}{dx} \right) dx$$

$$+ gH_{i+1,j}\zeta_{i,j}\int_{x_{i,j}}^{x_{i+1,j}} \left(\phi_{i+1,j}\frac{d\phi_{i,j}}{dx}\frac{d\phi_{i,j}}{dx} \right) dx + gH_{i+1,j}\zeta_{i+1,j}\int_{x_{i,j}}^{x_{i+1,j}} \left(\phi_{i+1,j}\frac{d\phi_{i+1,j}}{dx}\frac{d\phi_{i,j}}{dx} \right) dx$$

$$= gH_{i,j}\zeta_{i,j}\left(\frac{1}{2\Delta x}\right) + gH_{i,j}\zeta_{i+1,j}\left(\frac{-1}{2\Delta x}\right) + gH_{i+1,j}\zeta_{i,j}\left(\frac{1}{2\Delta x}\right) + gH_{i+1,j}\zeta_{i+1,j}\left(\frac{-1}{2\Delta x}\right)$$

$$= \frac{g}{2\Delta x}[(H_{i,j} + H_{i+1,j})(\zeta_{i,j} - \zeta_{i+1,j})] .$$

Fifth term of right element:

For the fifth term:

$$-\int_{x_{i,j}}^{x_{i+1,j}} \left(E_l \frac{\partial^2}{\partial x^2}(HU)\frac{d\phi_{i,j}}{dx} \right) dx = \int_{x_{i,j}}^{x_{i+1,j}} \left(E_l \frac{\partial^2 \zeta}{\partial t \partial x}\frac{d\phi_{i,j}}{dx} \right) dx \tag{61}$$

$$= \int_{x_{i,j}}^{x_{i+1,j}} \left(E_l \frac{\partial^2}{\partial t \partial x}(\zeta_{i,j}\phi_{i,j} + \zeta_{i+1,j}\phi_{i+1,j})\frac{d\phi_{i,j}}{dx} \right) dx$$

$$= \int_{x_{i,j}}^{x_{i+1,j}} \left(E_l \left(\frac{d\zeta_{i,j}}{dt}\frac{d\phi_{i,j}}{dx} + \frac{d\zeta_{i+1,j}}{dt}\frac{d\phi_{i+1,j}}{dx} \right) \frac{d\phi_{i,j}}{dx} \right) dx$$

$$= E_l\frac{d\zeta_{i,j}}{dt}\int_{x_{i,j}}^{x_{i+1,j}} \left(\frac{d\phi_{i,j}}{dx}\frac{d\phi_{i,j}}{dx} \right) dx + E_l\frac{d\zeta_{i+1,j}}{dt}\int_{x_{i,j}}^{x_{i+1,j}} \left(\frac{d\phi_{i+1,j}}{dx}\frac{d\phi_{i,j}}{dx} \right) dx$$

$$= E_l\frac{d\zeta_{i,j}}{dt}\left(\frac{1}{\Delta x}\right) + E_l\frac{d\zeta_{i+1,j}}{dt}\left(\frac{-1}{\Delta x}\right)$$

$$= \frac{E_l}{\Delta x}\left(\frac{d\zeta_{i,j}}{dt} - \frac{d\zeta_{i+1,j}}{dt} \right) .$$

Sixth term of right element:

For the sixth term:

$$\int_{x_{i,j}}^{x_{i+1,j}}\left(\tau HU\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx = \int_{x_{i,j}}^{x_{i+1,j}}((\tau_{i,j}\phi_{i,j} + \tau_{i+1,j}\phi_{i+1,j})((HU)_{i,j}\phi_{i,j} \tag{62}$$

$$+ (HU)_{i+1,j}\phi_{i+1,j})\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\Big)dx$$

$$= \tau_{i,j}(HU)_{i,j}\int_{x_{i,j}}^{x_{i+1,j}}\left(\phi_{i,j}\phi_{i,j}\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx + \tau_{i,j}(HU)_{i+1,j}\int_{x_{i,j}}^{x_{i+1,j}}\left(\phi_{i,j}\phi_{i+1,j}\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx$$

$$+ \tau_{i+1,j}(HU)_{i,j}\int_{x_{i,j}}^{x_{i+1,j}}\left(\phi_{i+1,j}\phi_{i,j}\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx + \tau_{i+1,j}(HU)_{i+1,j}\int_{x_{i,j}}^{x_{i+1,j}}\left(\phi_{i+1,j}\phi_{i+1,j}\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx$$

$$= \tau_{i,j}(HU)_{i,j}\left(\frac{-1}{3}\right) + \tau_{i,j}(HU)_{i+1,j}\left(\frac{-1}{6}\right)$$

$$+ \tau_{i+1,j}(HU)_{i,j}\left(\frac{-1}{6}\right) + \tau_{i+1,j}(HU)_{i+1,j}\left(\frac{-1}{3}\right)$$

$$= (HU)_{i,j}\left(\frac{-\tau_{i,j}}{3} + \frac{-\tau_{i+1,j}}{6}\right) + (HU)_{i+1,j}\left(\frac{-\tau_{i,j}}{6} + \frac{-\tau_{i+1,j}}{3}\right).$$

Seventh term of right element:

For the seventh term:

$$-\int_{x_{i,j}}^{x_{i+1,j}}\left(GHU\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx = -\int_{x_{i,j}}^{x_{i+1,j}}\left(G((HU)_{i,j}\phi_{i,j} + (HU)_{i+1,j}\phi_{i+1,j})\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx \tag{63}$$

$$= -G(HU)_{i,j}\int_{x_{i,j}}^{x_{i+1,j}}\left(\phi_{i,j}\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx - G(HU)_{i+1,j}\int_{x_{i,j}}^{x_{i+1,j}}\left(\phi_{i+1,j}\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx$$

$$= -G(HU)_{i,j}\left(\frac{-1}{2}\right) - G(HU)_{i+1,j}\left(\frac{-1}{2}\right)$$

$$= (HU)_{i,j}\left(\frac{G}{2}\right) + (HU)_{i+1,j}\left(\frac{G}{2}\right).$$

Pulling it all together:

Then the equation for the right element (Equation 54) can be written with a discretized right hand side as:

$$\left.\left(\frac{\partial Q}{\partial t}+GQ\right)\right|_{x_{i,j}} = \left[\frac{\mathrm{d}^2\zeta_{i,j}}{\mathrm{d}t^2}\left(\frac{\Delta x}{3}\right)+\frac{\mathrm{d}^2\zeta_{i+1,j}}{\mathrm{d}t^2}\left(\frac{\Delta x}{6}\right)\right]+\left[G\frac{\mathrm{d}\zeta_{i,j}}{\mathrm{d}t}\left(\frac{\Delta x}{3}\right)+G\frac{\mathrm{d}\zeta_{i+1,j}}{\mathrm{d}t}\left(\frac{\Delta x}{6}\right)\right] \tag{64}$$
$$+\frac{1}{\Delta x}[H_{i,j}U_{i,j}^2-H_{i+1,j}U_{i+1,j}^2]+\frac{g}{2\Delta x}[(H_{i,j}+H_{i+1,j})(\zeta_{i,j}-\zeta_{i+1,j})]$$
$$+\frac{E_l}{\Delta x}\left(\frac{\mathrm{d}\zeta_{i,j}}{\mathrm{d}t}-\frac{\mathrm{d}\zeta_{i+1,j}}{\mathrm{d}t}\right)+(HU)_{i,j}\left(\frac{-\tau_{i,j}}{3}+\frac{-\tau_{i+1,j}}{6}\right)+(HU)_{i+1,j}\left(\frac{-\tau_{i,j}}{6}+\frac{-\tau_{i+1,j}}{3}\right)$$
$$+(HU)_{i,j}\left(\frac{G}{2}\right)+(HU)_{i+1,j}\left(\frac{G}{2}\right),$$

which can be simplified slightly to become:

$$\left.\left(\frac{\partial Q}{\partial t}+GQ\right)\right|_{x_{i,j}} = \left[\frac{\mathrm{d}^2\zeta_{i,j}}{\mathrm{d}t^2}\left(\frac{\Delta x}{3}\right)+\frac{\mathrm{d}^2\zeta_{i+1,j}}{\mathrm{d}t^2}\left(\frac{\Delta x}{6}\right)\right] \tag{65}$$
$$+\left[\frac{\mathrm{d}\zeta_{i,j}}{\mathrm{d}t}\left(\frac{E_l}{\Delta x}+\frac{G\Delta x}{3}\right)-\frac{\mathrm{d}\zeta_{i+1,j}}{\mathrm{d}t}\left(\frac{E_l}{\Delta x}-\frac{G\Delta x}{6}\right)\right]$$
$$+H_{i,j}\left[\frac{U_{i,j}^2}{\Delta x}+\frac{g}{2\Delta x}(\zeta_{i,j}-\zeta_{i+1,j})\right]+H_{i+1,j}\left[\frac{-U_{i+1,j}^2}{\Delta x}+\frac{g}{2\Delta x}(\zeta_{i,j}-\zeta_{i+1,j})\right]$$
$$+(HU)_{i,j}\left(\frac{-\tau_{i,j}}{3}+\frac{-\tau_{i+1,j}}{6}+\frac{G}{2}\right)+(HU)_{i+1,j}\left(\frac{-\tau_{i,j}}{6}+\frac{-\tau_{i+1,j}}{3}+\frac{G}{2}\right).$$

Presentation of the Consistent Fluxes

We are trying to solve for the fluxes to the left and to the right of a node in a one-dimensional framework. These fluxes are the solution to the ordinary differential equations (ODEs) set forth in Equations 24 and 25, which are restated here for convenience:

$$\left.\left(\frac{\partial Q}{\partial t}+GQ\right)\right|_{x_{i,j-1}} = \int_{x_{i,j-1}}^{x_{i-1,j-1}}\left(\frac{\partial^2\zeta}{\partial t^2}\phi_{i,j-1}+G\frac{\partial\zeta}{\partial t}\phi_{i,j-1}-HU\frac{\partial G}{\partial x}\phi_{i,j-1}+A\frac{\mathrm{d}\phi_{i,j-1}}{\mathrm{d}x}\right)dx, \tag{66}$$

and:

$$\left(\frac{\partial Q}{\partial t} + GQ\right)\bigg|_{x_{i,j}} = \int_{x_{i,j}}^{x_{i+1,j}} \left(\frac{\partial^2 \zeta}{\partial t^2}\phi_{i,j} + G\frac{\partial \zeta}{\partial t}\phi_{i,j} - HU\frac{\partial G}{\partial x}\phi_{i,j} + A\frac{\mathrm{d}\phi_{i,j}}{\mathrm{d}x}\right)dx . \tag{67}$$

Both ODEs govern fluxes associated with node *i*, but Equation 66 represents the flux from the left (from element $j-1$), and Equation 67 represents the flux from the right (from element *j*). The right-hand sides of these two equations have been discretized in Equations 53 and 65, respectively. And, as stated above, these equations have the form:

$$\frac{\partial u}{\partial t} - au = f(t) , \tag{68}$$

which has the following analytical solution:

$$u = \int_0^t e^{a(t-s)} f(s) ds + e^{at} u_0 . \tag{69}$$

For our ODEs, we can integrate over a single time step, so the analytical solution becomes:

$$Q^{\Delta t} = \int_0^{\Delta t} e^{-G(\Delta t - s)} f(s) ds + e^{-G\Delta t} Q^t \tag{70}$$

$$= e^{-G\Delta t} \int_0^{\Delta t} e^{Gs} f(s) ds + e^{-G\Delta t} Q^t ,$$

where we have set the previous time $t = 0$. Then we approximate the integral using Huen's method:

$$Q^{\Delta t} = e^{-G\Delta t}\left\{\left(\frac{\Delta t}{2}\right)[e^{G\Delta t} f(\Delta t) + f(0)]\right\} + e^{-G\Delta t} Q^t , \tag{71}$$

where the quantity $f(\Delta t)$ is the discretized Equation 53 or 65, evaluated at time $t + \Delta t$, and the quantity $f(0)$ is the same discretized Equation 53 or 65, evaluated at time *t*. However, those equations contain time derivatives that must be evaluated at times t and $t + \Delta t$. We use the following difference approximations:

$$\frac{\partial^2 \zeta}{\partial t^2}\bigg|_t = \frac{\zeta^{t+\Delta t} - 2\zeta^t + \zeta^{t-\Delta t}}{(\Delta t)^2} , \tag{72}$$

$$\left.\frac{\partial\zeta}{\partial t}\right|_{t} = \frac{\zeta^{t+\Delta t} - \zeta^{t-\Delta t}}{2\Delta t}, \tag{73}$$

$$\left.\frac{\partial^2\zeta}{\partial t^2}\right|_{t+\Delta t} = \frac{2\zeta^{t+\Delta t} - 5\zeta^{t} + 4\zeta^{t-\Delta t} - \zeta^{t-2\Delta t}}{(\Delta t)^2}, \tag{74}$$

and:

$$\left.\frac{\partial\zeta}{\partial t}\right|_{t+\Delta t} = \frac{3\zeta^{t+\Delta t} - 4\zeta^{t} + \zeta^{t-\Delta t}}{2\Delta t}, \tag{75}$$

which are second-order accurate in time. Now we can finally state the equations for the fluxes on each side of a one-dimensional node.

For the element to the left of the node:

$$Q_{i,j-1}^{t+\Delta t} = \frac{(\Delta t)e^{-G\Delta t}}{2}\left(e^{G\Delta t}\left\{\left[\left(\frac{2\zeta_{i-1,j-1}^{t+\Delta t} - 5\zeta_{i-1,j-1}^{t} + 4\zeta_{i-1,j-1}^{t-\Delta t} - \zeta_{i-1,j-1}^{t-2\Delta t}}{(\Delta t)^2}\right)\left(\frac{-\Delta x}{6}\right)\right.\right.\right. \tag{76}$$

$$+\left(\frac{2\zeta_{i,j-1}^{t+\Delta t} - 5\zeta_{i,j-1}^{t} + 4\zeta_{i,j-1}^{t-\Delta t} - \zeta_{i,j-1}^{t-2\Delta t}}{(\Delta t)^2}\right)\left(\frac{-\Delta x}{3}\right)\Bigg] + \Bigg[\left(\frac{3\zeta_{i-1,j-1}^{t+\Delta t} - 4\zeta_{i-1,j-1}^{t} + \zeta_{i-1,j-1}^{t-\Delta t}}{2\Delta t}\right)\left(\frac{E_l}{\Delta x} - \frac{G\Delta x}{6}\right)$$

$$-\left(\frac{3\zeta_{i,j-1}^{t+\Delta t} - 4\zeta_{i,j-1}^{t} + \zeta_{i,j-1}^{t-\Delta t}}{2\Delta t}\right)\left(\frac{E_l}{\Delta x} + \frac{G\Delta x}{3}\right)\Bigg] + H_{i-1,j-1}^{t+\Delta t}\left[\frac{(U_{i-1,j-1}^{t+\Delta t})^2}{\Delta x} + \frac{g}{2\Delta x}(\zeta_{i-1,j-1}^{t+\Delta t} - \zeta_{i,j-1}^{t+\Delta t})\right]$$

$$+ H_{i,j-1}^{t+\Delta t}\left[\frac{-(U_{i,j-1}^{t+\Delta t})^2}{\Delta x} + \frac{g}{2\Delta x}(\zeta_{i-1,j-1}^{t+\Delta t} - \zeta_{i,j-1}^{t+\Delta t})\right]$$

$$\left.+ (HU)_{i-1,j-1}^{t+\Delta t}\left[\frac{-\tau_{i-1,j-1}}{3} + \frac{-\tau_{i,j-1}}{6} + \frac{G}{2}\right] + (HU)_{i,j-1}^{t+\Delta t}\left[\frac{-\tau_{i-1,j-1}}{6} + \frac{-\tau_{i,j-1}}{3} + \frac{G}{2}\right]\Bigg]\right\}$$

$$+\left\{\left[\left(\frac{\zeta_{i-1,j-1}^{t+\Delta t} - 2\zeta_{i-1,j-1}^{t} + \zeta_{i-1,j-1}^{t-\Delta t}}{(\Delta t)^2}\right)\left(\frac{-\Delta x}{6}\right) + \left(\frac{\zeta_{i,j-1}^{t+\Delta t} - 2\zeta_{i,j-1}^{t} + \zeta_{i,j-1}^{t-\Delta t}}{(\Delta t)^2}\right)\left(\frac{-\Delta x}{3}\right)\right]\right.$$

$$+\left[\left(\frac{\zeta_{i-1,j-1}^{t+\Delta t} - \zeta_{i-1,j-1}^{t-\Delta t}}{2\Delta t}\right)\left(\frac{E_l}{\Delta x} - \frac{G\Delta x}{6}\right) - \left(\frac{\zeta_{i,j-1}^{t+\Delta t} - \zeta_{i,j-1}^{t-\Delta t}}{2\Delta t}\right)\left(\frac{E_l}{\Delta x} + \frac{G\Delta x}{3}\right)\right]$$

$$+ H_{i-1,j-1}^{t}\left[\frac{(U_{i-1,j-1}^{t})^2}{\Delta x} + \frac{g}{2\Delta x}(\zeta_{i-1,j-1}^{t} - \zeta_{i,j-1}^{t})\right]$$

$$+ H_{i,j-1}^{t}\left[\frac{-(U_{i,j-1}^{t})^2}{\Delta x} + \frac{g}{2\Delta x}(\zeta_{i-1,j-1}^{t} - \zeta_{i,j-1}^{t})\right]$$

$$\left.\left.+ (HU)_{i-1,j-1}^{t}\left[\frac{-\tau_{i-1,j-1}}{3} + \frac{-\tau_{i,j-1}}{6} + \frac{G}{2}\right] + (HU)_{i,j-1}^{t}\left[\frac{-\tau_{i-1,j-1}}{6} + \frac{-\tau_{i,j-1}}{3} + \frac{G}{2}\right]\right\}\right)$$

$$+ e^{-G\Delta t}Q_{i,j-1}^{t}.$$

And, for the element to the right of the node:

$$Q_{i,j}^{t+\Delta t} = \frac{(\Delta t)e^{-G\Delta t}}{2}\left(e^{G\Delta t}\left\{\left[\left(\frac{2\zeta_{i,j}^{t+\Delta t} - 5\zeta_{i,j}^{t} + 4\zeta_{i,j}^{t-\Delta t} - \zeta_{i,j}^{t-2\Delta t}}{(\Delta t)^2}\right)\left(\frac{\Delta x}{3}\right)\right.\right.\right. \tag{77}$$

$$\left.+\left(\frac{2\zeta_{i+1,j}^{t+\Delta t} - 5\zeta_{i+1,j}^{t} + 4\zeta_{i+1,j}^{t-\Delta t} - \zeta_{i+1,j}^{t-2\Delta t}}{(\Delta t)^2}\right)\left(\frac{\Delta x}{6}\right)\right] + \left[\left(\frac{3\zeta_{i,j}^{t+\Delta t} - 4\zeta_{i,j}^{t} + \zeta_{i,j}^{t-\Delta t}}{2\Delta t}\right)\left(\frac{E_l}{\Delta x} + \frac{G\Delta x}{3}\right)\right.$$

$$\left.-\left(\frac{3\zeta_{i+1,j}^{t+\Delta t} - 4\zeta_{i+1,j}^{t} + \zeta_{i+1,j}^{t-\Delta t}}{2\Delta t}\right)\left(\frac{E_l}{\Delta x} - \frac{G\Delta x}{6}\right)\right] + H_{i,j}^{t+\Delta t}\left[\frac{(U_{i,j}^{t+\Delta t})^2}{\Delta x} + \frac{g}{2\Delta x}(\zeta_{i,j}^{t+\Delta t} - \zeta_{i+1,j}^{t+\Delta t})\right]$$

$$+ H_{i+1,j}^{t+\Delta t}\left[\frac{-(U_{i+1,j}^{t+\Delta t})^2}{\Delta x} + \frac{g}{2\Delta x}(\zeta_{i,j}^{t+\Delta t} - \zeta_{i+1,j}^{t+\Delta t})\right]$$

$$\left.+ (HU)_{i,j}^{t+\Delta t}\left(\frac{-\tau_{i,j}}{3} + \frac{-\tau_{i+1,j}}{6} + \frac{G}{2}\right) + (HU)_{i+1,j}^{t+\Delta t}\left(\frac{-\tau_{i,j}}{6} + \frac{-\tau_{i+1,j}}{3} + \frac{G}{2}\right)\right]\right\}$$

$$+\left\{\left[\left(\frac{\zeta_{i,j}^{t+\Delta t} - 2\zeta_{i,j}^{t} + \zeta_{i,j}^{t-\Delta t}}{(\Delta t)^2}\right)\left(\frac{\Delta x}{3}\right) + \left(\frac{\zeta_{i+1,j}^{t+\Delta t} - 2\zeta_{i+1,j}^{t} + \zeta_{i+1,j}^{t-\Delta t}}{(\Delta t)^2}\right)\left(\frac{\Delta x}{6}\right)\right]\right.$$

$$+\left[\left(\frac{\zeta_{i,j}^{t+\Delta t} - \zeta_{i,j}^{t-\Delta t}}{2\Delta t}\right)\left(\frac{E_l}{\Delta x} + \frac{G\Delta x}{3}\right) - \left(\frac{\zeta_{i+1,j}^{t+\Delta t} - \zeta_{i+1,j}^{t-\Delta t}}{2\Delta t}\right)\left(\frac{E_l}{\Delta x} - \frac{G\Delta x}{6}\right)\right]$$

$$+ H_{i,j}^{t}\left[\frac{(U_{i,j}^{t})^2}{\Delta x} + \frac{g}{2\Delta x}(\zeta_{i,j}^{t} - \zeta_{i+1,j}^{t})\right]$$

$$+ H_{i+1,j}^{t}\left[\frac{-(U_{i+1,j}^{t})^2}{\Delta x} + \frac{g}{2\Delta x}(\zeta_{i,j}^{t} - \zeta_{i+1,j}^{t})\right]$$

$$\left.\left.+ (HU)_{i,j}^{t}\left(\frac{-\tau_{i,j}}{3} + \frac{-\tau_{i+1,j}}{6} + \frac{G}{2}\right) + (HU)_{i+1,j}^{t}\left(\frac{-\tau_{i,j}}{6} + \frac{-\tau_{i+1,j}}{3} + \frac{G}{2}\right)\right]\right\}\right) + e^{-G\Delta t}Q_{i,j}^{t}.$$

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