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Impact of the Form of the Momentum Equation on Shallow Water Models Based on the Generalized Wave Continuity Equation

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A number of finite-element based shallow water models use the GWC (generalized wave continuity) algorithm, first introduced by Lynch and Gray in 1979 [1], to suppress short wave noise without artificial or numerical damping. Nearly all of these, including the ADCIRC (ADvanced 3D CIRCulation model developed by Westerink and Luettich, see [2]) model used in our research, utilize the non-conservative form of the momentum equation to obtain the velocity field. However, as early as 1990, it was discovered that the inconsistent treatment of the advective terms between the GWC equation (which uses the conservative form of the momentum equation in its formulation) and the non-conservative momentum (NCM) equation was causing instabilities. Additional analyses by Kinnmark [3] and Aldama [4], among others, provided further evidence that the conservative form of the momentum (CM) equation may improve accuracy and stability, particularly for simulations that have regions of highly advective flows, e.g., converging flow near inlets and around barrier islands. Furthermore, as the ADCIRC modeling group begins to explore alternative algorithms for the continuity equations (e.g., discontinuous Galerkin algorithm instead of GWC), the conservative form of the momentum equation would facilitate model coupling. For these reasons, we modified the ADCIRC code to use the conservative form of the momentum equation. The impact of this change on mass balance, stability, and accuracy (spatial and temporal) is rigorously assessed, first for 1D barotropic flows and later for 2D barotropic flows in a variety of basins. Results from the 1D experiments indicate the following: global mass balance improves up to two orders of magnitude, depending on the domain; local mass balance greatly improves in areas where steep bathymetry gradients occur; when combined with a predictor-corrector time-marching algorithm, stability increases by at least a factor of two (for a given level of mass balance error); and the form of the momentum equation does not cause a significant change in temporal or spatial accuracy when evaluating the two different time-marching algorithms.

1. INTRODUCTION

Several studies (e.g., [3-7]) have shown that GWC-based, finite element shallow water models may have problems conserving mass locally. Kinnmark [3] provided the first theoretical analysis of the mass conserving properties of the GWC equation. The GWC equation fits into the class of derivative equations, which can open up the solution space to include a wider range of solutions. In order to restrict the solution space of the GWC equation, auxiliary conditions must be satisfied. Kinnmark, through the use of operator notation, established the equivalence between the primitive form of the shallow water equations (continuity and conservative form of the momentum equations) and other formulations, including the wave continuity equation. For the GWC equation, he concluded that the continuity equation, including boundary conditions, must be exactly satisfied at start-up in order for mass to be conserved. However, because of roundoff errors and other noise that occurs during spin-up of a numerical model, this can not always be guaranteed. Also supporting this observation are Walters and Carey [8], who determined that the vanishing of the derivative of the continuity equation with respect to time alone is not sufficient to ensure mass conservation. Thus, Kinnmark found that it is necessary to meet one of the following conditions. 1) If the NCM equation is used, then $G > \nabla \bullet \mathbf{v}$, where G is the GWC numerical parameter and $\mathbf{v}$ is the velocity field. However, this can not always be guaranteed since $\nabla \bullet \mathbf{v}$ is not always known beforehand, in which case G should be chosen as large as possible without introducing spurious oscillations. 2) If the CM equation is used, then $G > 0$, which can always be guaranteed.

Studies done by Kolar et al. [5,7] in the early 1990's looked at the effects of this G parameter on mass conservation. One study [5] showed that the G parameter does indeed improve the mass conservation. Kolar et. al. [7] recommended using a G value between 1 to 10 times τ_{max}, where τ is nonlinear bottom friction, to create a balance between the wave continuity equation and the primitive form of the equations. Boundary conditions were analyzed by Kolar et al. [6] and Lynch and Holboke [9] to determine if implementing them differently would improve mass conservation. They found that properly implemented boundary conditions could eliminate global mass balance errors; however, they did not eliminate the local mass balance errors. Kolar et al. [7] also looked at recasting the convective acceleration terms in the GWC into non-conservative form, so as to mimic the form in the momentum equation. This reformulation improved both global and local mass conservation but did not eliminate the problem. Also, it introduced both time and space derivatives into the convective (advective) terms.

In 2000, Aldama et al. [4] analyzed the mass conservation ability of the GWC and the NCM equations in the continuous forms, using both a Taylor-Frechet and a Fourier series analysis. They showed that the derivative formulation of the GWC is not consistent with the mass conservation principle. Also, an analysis of the equations in the discrete form showed that with increasing values of G, the mass conservation performance of the GWC equation improves.

Improvements in grid generation techniques can also provide gains in accuracy and stability. Hagen et al. [10] developed a new technique that looked at local truncation errors associated with the linearized, non-conservative form of the momentum equation. They determined that grid refinement in areas where these errors are large, (e.g., in areas where steep bathymetry gradients occur) can improve the overall accuracy of the solution, without an increase in computational burden. Not coincidently, these areas correspond to where the velocity-based NCM solution changes rapidly.

In summary, then, we observe the following about NCM-based GWC models: 1) local mass balance errors and instabilities can occur, particularly in regions with highly advective flows; 2) numerical and analytical studies demonstrate that the problems can be mitigated (but not eliminated) by proper choice of G, by reformulating the advective terms, and by proper treatment of the boundary conditions; and 3) high levels of grid refinement are needed in areas with steep bathymetry gradients to minimize truncation errors. Based on these observations, we hypothesize that changing to the conservative form of the momentum equation, which is flux, not velocity-based, may improve both global and local mass conservation, eliminate the need to reformulate the convective terms between the governing equations, and lessen the need for extensive refinement in areas with steep bathymetry gradients. Also, use of the conservative form of the momentum equation makes it more natural to bring in flux boundary conditions and facilitates coupled models (e.g., discontinuous and continuous Galerkin methods). The primary objectives of this paper, then, are to assess the impact of the conservative form of the momentum equation on mass conservation, stability, temporal and spatial accuracy, first for the 1D version of ADCIRC, and subsequently the 2D version.

2. BACKGROUND AND EQUATION CHANGES

Initial testing of the conservative momentum (CM) equation uses the 1D version of ADCIRC. Two additional assumptions for this study are that contributions from the atmospheric forcing are negligible and eddy viscosity is constant. In 1D, the GWC and the CM equations are as follows:

$$\frac{\partial^2\zeta}{\partial t^2} + G\frac{\partial\zeta}{\partial t} - q\frac{\partial G}{\partial x} - \frac{\partial}{\partial x}\left[\frac{\partial(qu)}{\partial x} + (G-\tau)q + gH\frac{\partial\zeta}{\partial x} - \varepsilon\frac{\partial^2 q}{\partial x^2}\right] = 0 \tag{1}$$

$$\frac{\partial(q)}{\partial t} + \frac{\partial(qu)}{\partial x} + \tau q + gH\frac{\partial\zeta}{\partial x} - \varepsilon\frac{\partial^2(q)}{\partial x^2} = 0 \tag{2}$$

where $Hu = q$ is the flux, u is the depth-averaged velocity, τ is the nonlinear bottom friction, ε is the eddy viscosity, t is time, ζ is the elevation changes, x is the distance, and $H = h + \zeta$ is total water column depth. These equations use finite elements for the spatial discretization, while for the temporal discretization, a three time-level scheme centered at k is used in (1) and a two time-level scheme centered at $k + 1/2$ is used in (2). In the original formulation, we allow for unlumped or lumped matrices to occur in the solutions; however, we now also allow for partial lumping. Solutions are obtained by first solving (1) for the elevation changes and then using (2) for the velocity results.

3. NUMERICAL EXPERIMENTS AND RESULTS

Four domains were used to evaluate the momentum changes: a constant bathymetry at a 5 meter depth; a parabolic bathymetry, which has a rate of change that varies as a 2nd order polynomial (Figure 1a); a Western North Atlantic bathymetry (Figure 1b, denoted East Coast); and a sinusoidal bathymetry (Figure 1c). The domains were chosen based on their ability to mimic 2D applications. Maximum and minimum depth values for the parabolic bathymetry were 300 and 3 meters, for the sinusoidal bathymetry - 200 to 2 meters, while for the East Coast bathymetry - 5000 to 20 meters. Boundary conditions on each of the domains were a 1 meter M_2 tide at the ocean boundary and no normal flux at the land boundary. Each was discretized with both constant and variable nodal spacing.

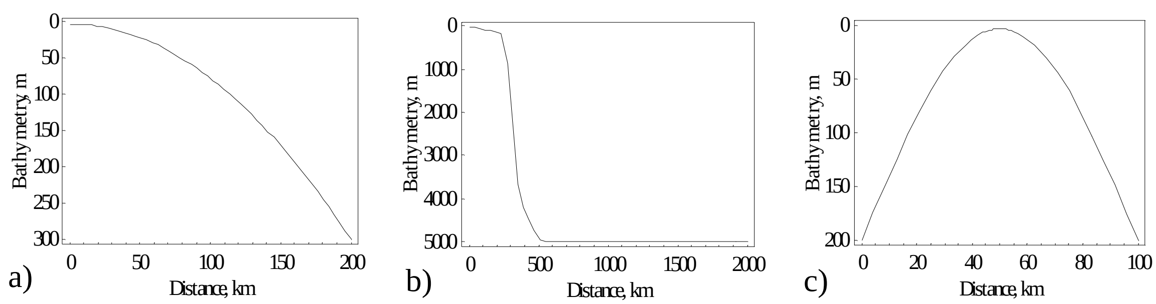

Figure 1. Schematics of the 1D domains. a - Quadratic, b - East Coast and c- Sinusoidal

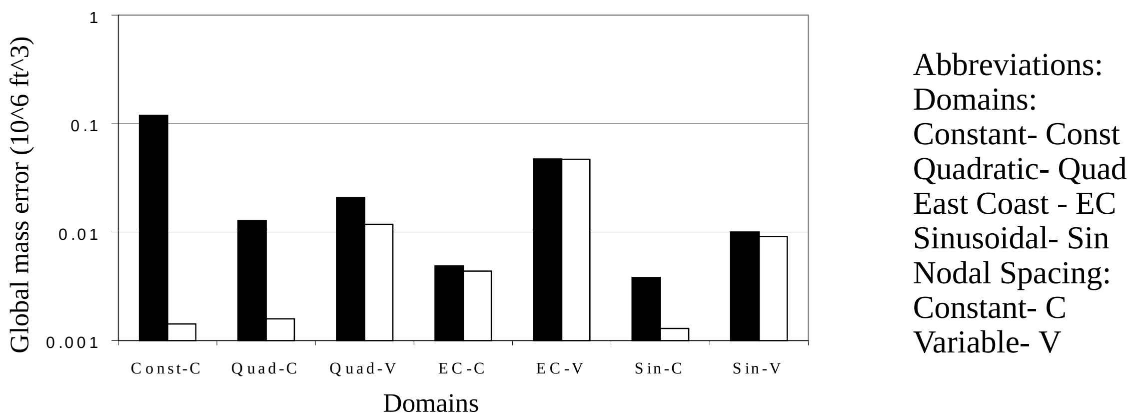

Figure 2. Errors in the global mass conservation for all the domains; NCM equation (solid bars), CM equation (open bars).

3.1. Mass Balance

In order to evaluate the mass balance characteristics of the two momentum equation formulations, we compared the mass accumulations with cumulative net flux for each region, both locally and globally [5]. Figure 2 shows the errors in the global mass conservation for the NCM and CM equations for all domains. We evaluated two node spacings, constant and variable, denoted C and V, respectively, in Figure 2. All parameter values are the same within each domain, and the advective terms in the NCM equation were left in native form. Results show the change to the CM equation improves global mass balance in all cases. In most of the domains, we obtain up to a two order of magnitude improvement, except for the East Coast domain. The similar behavior may be due to the amount of nodes in the deeper bathymetry where advective terms are small. As for the local mass conservation results, we looked at two domains that had the steepest bathymetry gradients: the East Coast (Figure 1b) and the sinusoidal (Figure 1c). Both cases used a variable nodal spacing. Local mass balance was analyzed over the simulation for each element and reported as an absolute mass error; results are shown in Figure 3. A schematic of the bathymetry for each domain is also shown in the figure by the longer dashed line. As can be seen, the CM formulation provides significant improvements in areas where there is a steep bathymetry gradient. In Figure 3, note that the open boundary (node 0) shows larger local mass balance errors than the land boundary. Finally, we note that different lumping parameters were examined in these experiments; however, results showed that changing the lumping parameters did not significantly impact the results. Therefore, we used the original lumping parameters of ADCIRC (GWC equation not lumped and the NCM or CM equation lumped) in subsequent experiments.

3.2. Convergence Testing

For the convergence tests, we used the NCM equation with changes to the advective terms and the CM equation. In order to compare the momentum equations and the convergence properties, we looked at the average L_2 norm between the two formulations over a simulation. Ideally, we should see less distinction between the two equations as we refine the grid since we are approaching the (same) “true” solution. In these experiments, we analyzed three domains: the quadratic (both constant and variable spacing), the East Coast (constant spacing) and the sinusoidal (constant spacing). These results (both velocity and elevation) indicate that, as the

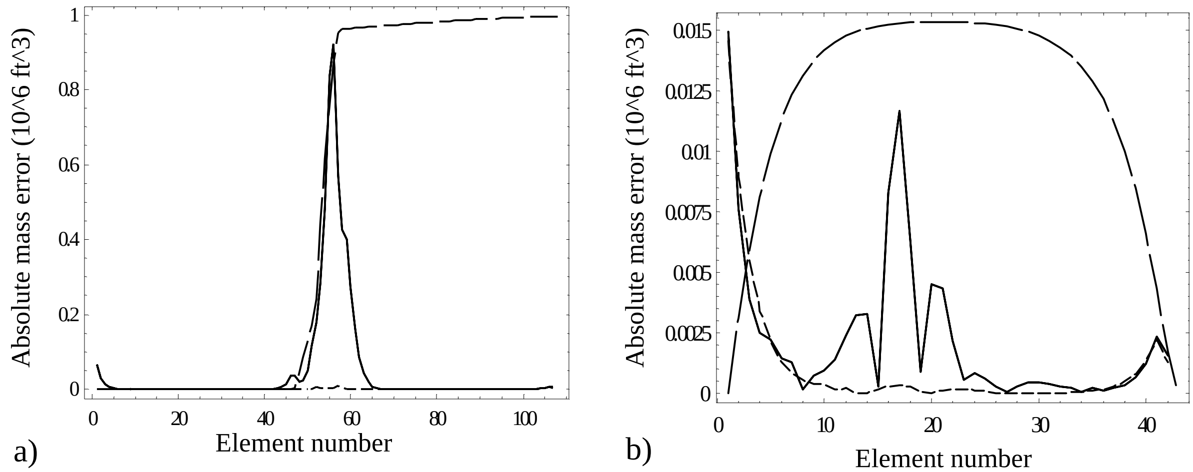

Figure 3. Local mass conservation results for two formulations of the momentum equation for a) East Coast, b) Sinusoidal. (Longer dashes - bathymetry (not to scale), solid line - NCM, and long dashes - CM)

grid is refined, the two formulations of the momentum equation results converge.

3.3. Stability

In the stability tests, we examined the two formulations of the momentum equation and determined which allowed for the greater time step. Experiments in this section utilized all four domains with either constant or variable nodal spacing. No change in the maximum allowable time step was observed. The only exception to this result was the sinusoidal domain with variable spacing, in which we saw a decrease in the maximum allowable time step. In general, we find that the CM maintains the same stability as the NCM equation.

3.4. Accuracy (Temporal and Spatial)

For the accuracy experiments, both temporal and spatial, we analyzed both formulations of the momentum equation by looking at the average L_2 norm over the simulation. A "true" (fine grid and time step) solution is established for each domain. Each of the domains were analyzed using either constant or variable nodal spacing, as noted. Figure 4 shows the temporal accuracy results with the spacing, either constant or variable, and type of bottom friction parameterization indicated in the figure caption. Note that these results are only for the elevation changes; however, velocity results show similar trends. Results indicate that the NCM equation has lower absolute errors than the CM equation in all of the domains analyzed, but most of the results showed only a minor difference in the errors. For each domain, we find that the order of accuracy (convergence rate) for each formulation is nearly the same. Figure 5 shows the spatial accuracy results with the spacing and type of the bottom friction parameterization indicated in the figure caption. Results show that the different formulations of the momentum equation does not significantly affect the spatial accuracy.

3.5. Predictor-Corrector Time-Marching Algorithm

Another aspect of this work looked at the predictor-corrector time-marching algorithm, discussed in Kolar et al. [11] and Dresback et al. [12]. This change allows for implicit treatment of the nonlinear terms. We again compare the two formulations of the momentum equation (CM

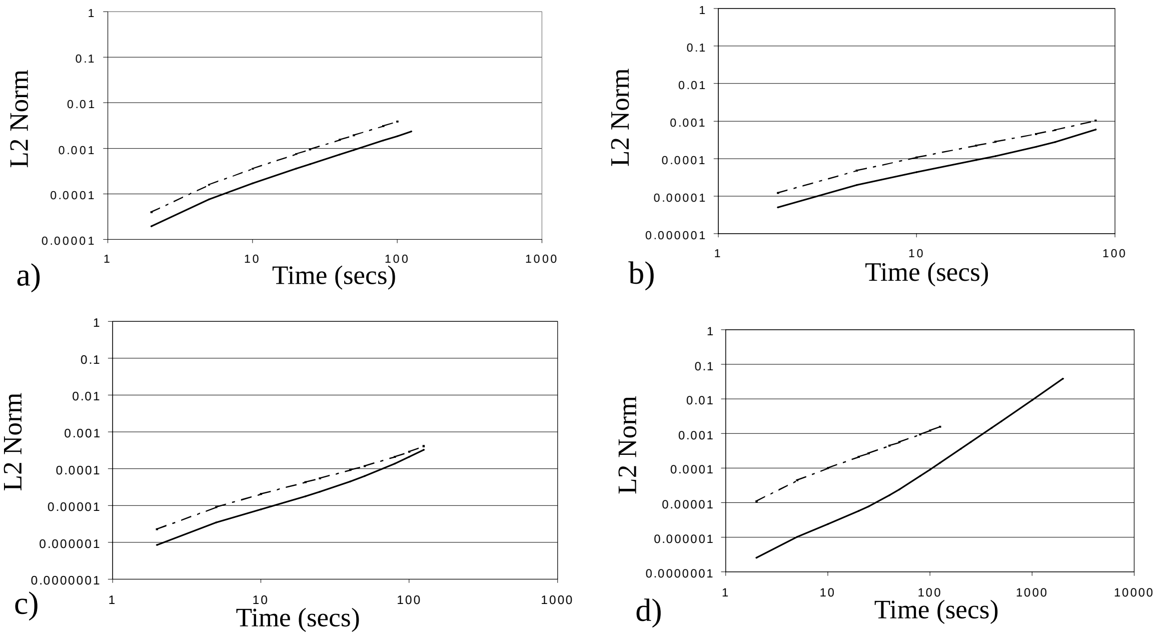

Figure 4. Temporal accuracy results (elevation) for all four domains: a) constant (constant), b) quadratic (variable), c) East Coast (variable) and d) sinusoidal (variable). Solid line - NCM, Long dashes - CM. (Note: first three results used a constant bottom friction while the last result used a variable bottom friction.)

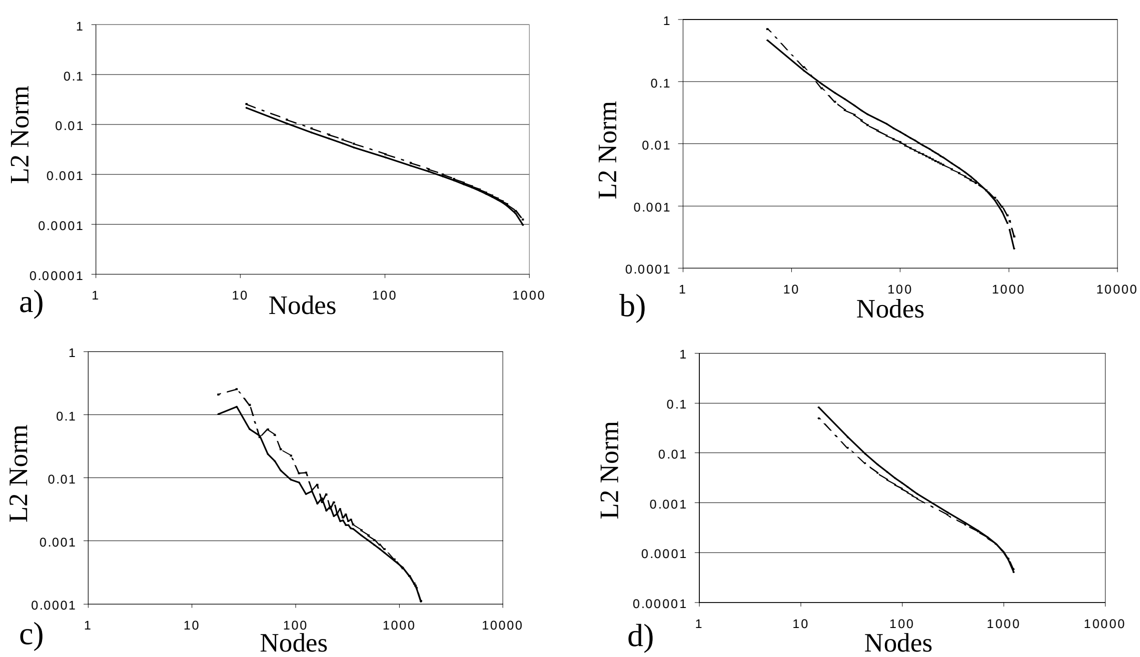

Figure 5. Spatial accuracy results (elevation) for all four domains: a) constant (constant), b) quadratic (variable), c) East Coast (variable) and d) sinusoidal (variable). Solid line - NCM, Long dashes - CM. (Note: first three results used a constant bottom friction while the last result used a variable bottom friction.)

vs. NCM) in conjunction with the predictor-corrector algorithm for changes in mass balance, stability, and temporal and spatial accuracy. The stability and mass balance were analyzed together to obtain the maximum allowable time step without increasing the global mass balance error from the original time-marching algorithm (at its maximum time step). In this experiment, we find that the maximum allowable time step with the CM equation is equal to or slightly less than the one using the NCM equation. However, all of the results meet the required two-fold increase necessary to make the predictor-corrector time-marching algorithm cost-effective. In all cases, this two-fold increase was obtained while still maintaining nearly the same global mass balance errors, as found with the original time-marching algorithm. As for the temporal accuracy results, we find that the two formulations provide similar results as in Section 3.4, but the conservative formulation provides slightly better results in some parts of the domains. This is different from the original time-marching algorithm. Spatial accuracy follows the behavior of the original time-marching algorithm.

3.6. 2D Preliminary Results

The CM equation changes have also been implemented in the 2D version of ADCIRC. Initial experiments in 2D looked at the effect on stability with both the original time-marching algorithm and the predictor-corrector time-marching algorithm. These results show a decrease in the maximum allowable time step by 10-40% with the CM equation in conjunction with the predictor-corrector time-marching algorithm, and a less than 10% change with the CM equation in conjunction with the original time-marching algorithm. A sampling of local mass balance errors for the Bahama domain shows that the CM equation improves results in 70% of the elements. Global mass conservation has not yet been analyzed but is part of the current ongoing studies, along with temporal and spatial accuracy.

4. CLOSING OBSERVATIONS

In summary, the use of the CM equation in place of the NCM equation leads to the following observations:

- Use of the CM equation improves global mass conservation for all of the domains analyzed, with most cases showing a *decrease* of two orders of magnitude in the errors. As importantly, the CM equation *greatly improves* local mass conservation in areas with steep bathymetry gradients.
- Stability, temporal and spatial accuracy results did not change significantly between the two formulations of the momentum equation.
- Combining the CM equation with the predictor-corrector time-marching algorithm shows the necessary two-fold improvement in maximum allowable time step for stability, while still maintaining the global mass conservation results with the original time-marching algorithm. Spatial accuracy does not show a significant change. Temporal accuracy results are similar between the two formulations of the momentum equation with improvements in some parts of the domains.

Basically, the 1D experiments show that use of the CM equation produces global and local mass balance errors that are significantly less than the NCM equation, while other aspects of the model (i.e., stability and convergence rates) remain similar between the two formulations. The improvement in mass conservation is enough, in itself, to warrant continuation of 2D studies.

5. ACKNOWLEDGEMENTS

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