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Truncation Error Analysis of Shallow Water Models Based on the Generalized Wave Continuity Equation

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Abstract

Finite element solution of the shallow water wave equations has found increasing use by researchers and practitioners in the modeling of oceans and coastal areas. Wave equation models successfully eliminate spurious oscillation modes without resorting to artificial or numerical damping. Since its introduction in 1979, the wave equation algorithm has been modified in a number of substantial ways. Three changes were introduced to improve the accuracy of the algorithm, especially with respect to mass balance errors and errors in the generation of nonlinear constituents: 1) the convective terms in the wave equation were reformulated so as to have the same form as the advective terms in the momentum equation; 2) the primitive continuity equation was weighted with a numerical parameter whose magnitude is larger than the bottom friction; 3) normal fluxes on the boundary were treated as natural conditions with the flux interpreted as external to the computational domain. In previous work, we have provided a theoretical basis for the second and the third modification. However, the first was developed in a purely heuristic manner. In the present work, we use truncation error analysis to provide a theoretical understanding of how this algorithmic change produces such a dramatic improvement in computed results. It is shown that when the form of the advective terms in the wave continuity and momentum equation is the same, global truncation error is reduced by 78% for the model problem. Truncation error analysis, as implemented herein, is a very general and powerful tool that will lead to further insights and improvements of the wave continuity algorithm.

Background

Shallow water equations, based on conservation of mass and momentum, describe the propagation of long water waves in oceans, estuaries, and impoundments. Early attempts to solve the equations using the finite element method were frequently plagued by spurious oscillations superimposed on the true solution. Lynch and Gray[1] and Gray[2] introduced the wave continuity equation in 1979 as a means to successfully suppress the numerical noise without resorting to numerical or artificial damping of the solution. Since the inception of the formulation, the original algorithm has been modified in a number of substantial ways: 1) a numerical parameter was introduced to provide a more general means of weighting the primitive continuity equation[3]; 2) viscous dissipation terms were incorporated[4,5]; 3) three-dimensional simulations were realized by resolving the velocity profile in the vertical[6,7]; 4) interpolation of the advective (also called convective) terms was modified[6]; 5) algorithms were optimized for scalar and vector computers[6]; and 6), most recently, the method of implementing boundary conditions was altered[8]. The resulting algorithm has been extensively tested using analytical solutions and field data and is being used to model the hydrodynamic behavior of coastal and oceanic areas[9-11].

Modifications numbered 1, 4, and 6 were driven by the need to improve mass conservation in highly nonlinear applications, which, we have found, serves as a surrogate variable for errors in the generation of nonlinear constituents and phasing errors. Of these, numbers 1 and 6 have a sound theoretical basis. In Kolar et al.[12], dispersion analysis is used to study what effect G/τ ratios greater than one (where G is the numerical parameter in the wave continuity equation and τ is the bottom friction) have on the solution. A new method of implementing boundary conditions (modification 6) was derived from first principles using generalized functions in the finite element formulation[8]. In reference 12, we provide *empirical* evidence that reformulating the convective terms in the generalized wave continuity equation so that a consistency exists between the momentum and continuity equations (modification 4) greatly improves mass conservation. Herein, we use truncation error analysis to provide a theoretical understanding of the advective term modification. However, the analysis is not restricted to just analyzing the behavior of the advective terms. Rather, it is a very powerful analysis tool that allows us to look inside the wave continuity algorithm with the intent to better understand its numerical characteristics. For example, in Hagen et al.[13], truncation error analysis is used to optimally place nodes in the generation of finite element meshes. Ultimately, results from a complete study of truncation error will lead to improved accuracy, stability, and efficiency of the wave equation algorithm.

Conservation Equations

Primitive forms of the balance laws are obtained by vertical averaging of the microscopic balance laws. In the interest of brevity, the equations will be presented using operator notation; the full equations can be found in a variety of sources, including previous CMWR conference proceedings[6,12,14,15]. Let L represent the primitive continuity equation, $\mathbf{M}$ the non-conservative form of the momentum equation (NCM), and $\mathbf{M}^c$ the conservative form of the momentum equation. The generalized wave continuity equation (GWC) is obtained from

$$W^G \equiv \frac{\partial L}{\partial t} + GL - \nabla \bullet \mathbf{M}^c = 0 \tag{1}$$

where G is a numerical parameter. The wave continuity equation, as it originally appeared in Lynch and Gray[1], is obtained by setting $G = \tau$ where τ is the bottom friction parameter. Note that the primitive continuity equation can be viewed as a limiting form of the GWC equation by letting $G \to \infty$.

Discrete Equations

Discrete equations are obtained by interpolating W^G and $\mathbf{M}$ with C^0 linear finite elements. Implicit time discretization of W^G uses a three time level approximation centered at k. Time discretization of $\mathbf{M}$ uses a lumped two time level approximation centered at $k + 1/2$; the equations are linearized by formulating the advective terms explicitly. Exact quadrature rules are employed. A time-splitting solution procedure is adopted wherein W^G is first solved for nodal elevations and $\mathbf{M}$ is then solved for the velocity field. The resulting discrete equations can be found in Luettich et al.[6]

Truncation Error Analysis

Procedure

Truncation error analysis is based on Taylor's theorem, which allows all dependent variables to be evaluated at a common point, essentially the calculus equivalent of the arithmetic operation of finding a common denominator. Taylor's theorem in one dimension states that for any smooth function, i.e., a continuous function whose first N derivatives are continuous:

$$f(x) = f(x_0) + (x - x_0)\frac{df}{dx} + \frac{(x - x_0)^2}{2!}\frac{d^2 f}{dx^2} + \frac{(x - x_0)^3}{3!}\frac{d^3 f}{dx^3} + R^{N+1} \tag{2}$$

where x_0 is fixed point in the neighborhood of x at which the derivatives in (2) are evaluated, and R^{N+1} is the remainder term[16]. For the index notation encountered with discrete equations, a more workable form of Taylor's theorem is

$${}_{j+1} = f_j + \Delta x_j \frac{df}{dx} + \frac{\Delta x_j^2}{2!}\frac{d^2 f}{dx^2} + \frac{\Delta x_j^3}{3!}\frac{d^3 f}{dx^3} + R^{N+} \tag{3}$$

where all derivatives on the right hand side of (2) are evaluated at x_j. Multi-dimensional forms of (2) follow analogously.

Truncation error is found by first substituting in the Taylor series expansion for the dependent variables in the discrete equations so that all functions are evaluated at a common node, typically taken as (j, k) in two dimensions. The result is then subtracted from the continuous equation to generate the truncation error. If C represents the continuous differential equation and A the discrete approximation, then the truncation error (TE) is defined as

$$TE \equiv C - A \tag{4}$$

For practical reasons, TE analysis was historically restricted to linear problems in one or two dimensions. Three-dimensional and nonlinear problems were seldom analyzed because the resulting expansions become unwieldy and the manipulations prone to errors, especially when multiplying high order Taylor series in nonlinear cases. If attempted at all, they were typically limited to the lowest order terms on a uniform grid. However, with the advent of symbolic manipulators, even the most complex problems can be scrutinized using truncation error analysis, as long as the discrete equations can be written down explicitly. Moreover, nonuniform grids and expansions of any order can be studied. (We should qualify this as *reasonable* order since we still have to deal with the limits of computer memory, although this is becoming less of a handicap.) In the present study, all analyses were carried out to sixth order terms of the Taylor series expansion using *Mathematica*®.

Qualitative evaluations of the error expressions can give information about consistency and accuracy. However, quantitative analysis is precluded because the sign and magnitude of the derivative terms in (4) are unknown. Consequently, to quantify the truncation error, we set up a model problem and generated a fine grid solution with known convergence properties[8]. Output from the fine grid solution (elevation and velocity field) is fed into central difference formulas to approximate the derivatives in (4) at all interior points of the domain. From that point, parameter values can be defined and the truncation error quantified.

Truncation error analysis, as described above, is inherently an analysis of local error around a grid point. However, global error can be obtained by moving the evaluation point (j, k) around the domain of interest. For this work, global truncation error is defined as the average of the absolute value of the local truncation error as the analysis point is moved around the mesh, i.e.,

$$TE_{\text{global}} \equiv \frac{\sum_{j=1}^{N_j}\left[\sum_{k=1}^{N_k}\left|TE_{j,k}\right|\right]}{N_j + N_k} \qquad \textbf{(5)}$$

where $TE_{j,k}$ represents the local truncation error computed from (4) and N_j and N_k represent the number of mesh points in the j, k directions, respectively.

Model Problem

A shallow one-dimensional channel was used for the model problem so that significant nonlinear components are generated; a simulation scenario that mimics the errors seen in two-dimensional models of shallow coastal areas. Conditions for the problem are:

channel coordinates	$0 \le x \le 50$ km
channel depth	5 m
eddy viscosity ε	0.000 m^2/sec
$\Delta t_{\text{fine grid}}$	4 sec
$\Delta x_{\text{fine grid}}$	0.5 km
boundary conditions	$\zeta(0, t) = 1.0\sin[2\pi t/12.42 \text{ hrs}]$ m
	$u(50, t) = 0.0$ m/sec
initial conditions	cold start: $\zeta(x, 0) = u(x, 0) = 0.0$

bottom friction τ	0.0001 sec^{-1} (constant)
Courant number	0.056
$\Lambda_{M2}/\Delta x$	626

where in the last line, Λ_{M2} is the wavelength of the M2 wave. Note the fine temporal and spatial resolution. The x-axis is defined as positive to the east. The boundary conditions describe a channel with a land boundary at $x = 50$ km being forced by an M2 tide with 1 meter amplitude at $x = 0$ km. The simulation is spun up from a cold start and allowed to run for 24 tidal cycles; results from the last tidal cycle are saved for input to the truncation error equations.

A number of numerical parameters exist in the code; they are summarized below along with their default value. Physical parameters such as τ and ε were not adjusted during the truncation error study.

G	0.001 sec^{-1}	relative weight of L; see equation (1)
α_1	0.35	amount $\mathbf{g}\nabla\zeta$ term in GWC weighted at $k+1$, $k-1$
α_2	0.30	amount $\mathbf{g}\nabla\zeta$ term in GWC weighted at k
flag_gwc	0	determines form of GWC advective term; see below
flag_ncm	0	determines form of NCM advective term; see below
β_1	0.5	amount viscous term in NCM weighted at $k+1$
β_2	0.5	amount viscous term in NCM weighted at k
λ	0	lumping parameter for mass matrices; 0 is consistent mass matrix, 1 is fully-lumped mass matrix

Note that $2\alpha_1 + \alpha_2 = 1$ and $\beta_1 + \beta_2 = 1$.

Results - Qualitative

Expressions for the leading error terms of the full nonlinear, interior equations of W^G and $\mathbf{M}$ are quite lengthy and cannot be shown here. (They are available from the first author upon request.) Discrete equations at the boundary are of a different form and are not included in this analysis. In the expansions, we assume a constant time step, Δt, but allow for variable grid spacing.

Order of Accuracy

Analysis of the symbolic TE expressions shows that W^G is first order accurate in space if $\Delta x_j \neq \Delta x_{j+1}$, and it is second order accurate if $\Delta x_j = \Delta x_{j+1}$. If flag_gwc $\neq 0$, then W^G is first order accurate in time; if flag_gwc $= 0$, then W^G is second order accurate in time. Similarly, $\mathbf{M}$ is formally first order accurate in time, and first order in space unless $\Delta x_j = \Delta x_{j+1}$ in which case it is second order accurate.

Consistency

If we take the limit of the symbolic error expressions as the space/time mesh is reduced to zero, then $TE \rightarrow 0$ regardless of parameter values. Thus, the algorithm is consistent with the differential equation. To the authors' knowledge, this is the first time since the inception of the algorithm in 1979 that the full nonlinear GWC algorithm (based on the discretization scheme outlined earlier) has been proven to be consistent.

Results - Quantitative

Recall that for this presentation, we are primarily focusing on the behavior of the truncation error with respect to the advective terms. In the solution algorithm, a flag (a numerical parameter) determines the form of the advective terms in W^G and $\mathbf{M}$. If the flag is set to 1, then the advective term is in non-conservative form. If the flag is set to 0, then the advective term is in conservative form. Past numerical experiments[12] have shown that the most accurate results, in terms of mass conservation and constituent error, are obtained when the flags for W^G and $\mathbf{M}$ have the same value, i.e., both 1 or both 0. Of course, the flag may take on fractional values, which is also analyzed using the truncation error program.

Global Truncation Error

When the global truncation error formula (equation (5)) is applied to the model problem, we find that the truncation error is reduced by 78% when the advective term flag in W^G is changed from 0 (**not** the same as the flag in $\mathbf{M}$) to 1 (the same as the flag in $\mathbf{M}$). The result is consistent with our empirical results in reference 12.

Local Truncation Error

Locally, we see the same reduction in truncation error as the flag for the advective term is changed from 0 to 1. Figure 1 displays the local truncation error, as defined by equation (4), versus time (x is held constant at 1830 m) for various values of the flag in W^G. As can be seen, for nearly all times the truncation error decreases as the flag changes from 0 to 1. The two regions where there is virtually no change in truncation error occur at slack tide. At slack tide the velocity, and hence the advective term, is zero; since flag_gwc only affects the advective terms, TE is independent of the flag's value at slack tide.

Not shown is the surface plot of the entire truncation error versus space and time, but analysis of the plot reveals that most of the TE reduction occurs near the open ocean boundary ($x = 0$), the region of maximum error in the simulation. Little or no reduction takes place near the land boundary.

In summary, looking at the quantitative TE for the GWC equation, which is of primary concern when trying to improve mass conservation, we find that the truncation error is minimum when the flag for W^G is the same as the flag for $\mathbf{M}$. Before this analysis, we hypothesized that by having the advective terms of the same form, one would see some cancelling of truncation error terms between the continuity and momentum equation. However, analysis of both the symbolic TE expressions and the numerical values found from (5) shows that the truncation error associated with the $\mathbf{M}$ equation behaves somewhat independently of the W^G error. Moreover, there is an order of magnitude difference between the error terms. Thus we are led to believe that the improved accuracy is not caused by self-cancelling errors; rather, it is the outcome of an improved approximation of elevation and velocity by the respective equations when the flags are set to the same value.

Summary

Truncation error analysis, when done in conjunction with symbolic manipulators, allows researchers to analyze algorithmic behavior of complex nonlinear

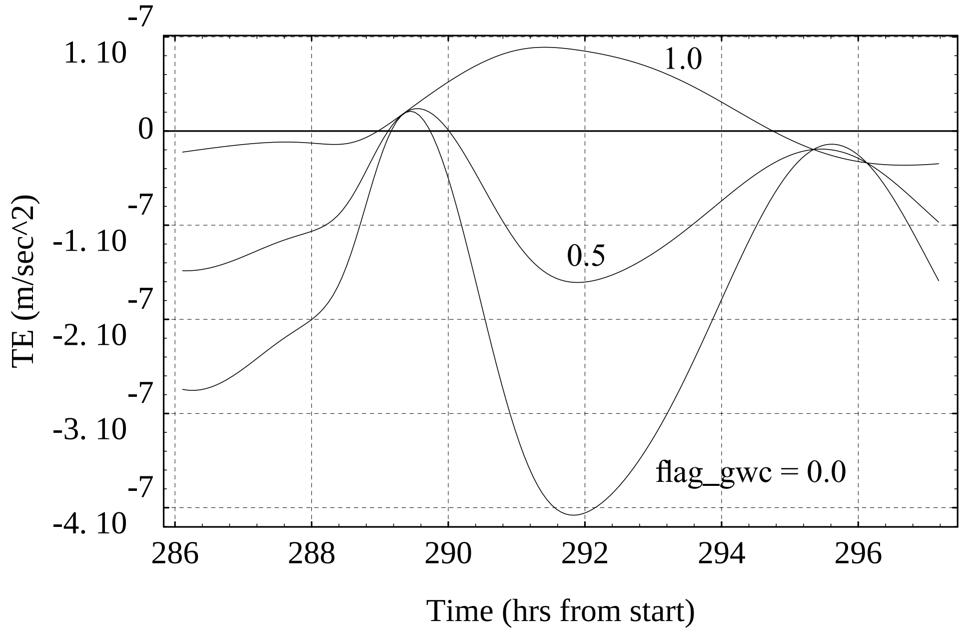

Figure 1 Behavior of truncation error of the GWC equation versus time for various values of flag_gwc. The x-coordinate is fixed at 1830 m (near the open ocean boundary). Note the decrease in truncation error as flag_gwc goes from 0 to 1.

problems on variable grids. Herein, we use the method to analyze the behavior of the truncation error in the wave continuity algorithm as the form of the advective term is varied. Results confirm output from previous numerical experiments, viz, the truncation error is minimized when the form of the advective term between the GWC and NCM equations is the same (e.g., advective terms in both equations in conservative form). On-going truncation error and stability studies will lead to new insights and algorithmic improvements of shallow water models based on the wave continuity equations.

Key Words: truncation error, shallow water models, wave continuity equation, finite elements, symbolic manipulators

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