Phase behavior of a finite volume shallow water algorithm

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ABSTRACT: Two algorithms that have been used for shallow water simulations on unstructured meshes are the generalized wave continuity (GWC) algorithm and selective mass lumping (SLFE). Both produce noise free solutions; the former by its ability to propagate short wave components, and the latter by its ability to damp the short waves that are not propagated. Recently, the finite volume method, which has been used in gas dynamics, has emerged as the basis for another solution algorithm on unstructured meshes. Advantages include its ability to conserve mass at an element level and its ability to handle shocks in the solution (supercritical flow problems). Herein we demonstrate that the phase propagation behavior, as represented by Fourier analysis, of the low-order finite volume method is more similar to SLFE than GWC. On the other hand, with the proper choice of parameters, the dispersion relation for the finite volume method is more similar to the GWC than SLFE. Thus, the algorithm can uniquely resolve all frequencies with minimal damping.

1 INTRODUCTION

1.1 *Motivation*

Shallow water equations (SWE) are used to model hydrodynamics in lakes, estuaries, coastal regions, and other impoundments. Accurate solutions demand that the algorithm does not introduce spurious modes. Examples of such solutions include the wave continuity approach (Lynch and Gray 1979) and the staggered finite difference approach (Leendertse 1967). A common characteristic of these solutions is a monotonic dispersion relationship. Fourier analysis can also be used to examine the phase propagation and damping properties of a discrete algorithm, where the desired behavior is to have only those waves that are out of phase be damped in the solution process (typically high frequency components on the order of $2\Delta x$).

A new solution method that has recently come onto the scene in shallow water applications is the finite volume method (FVM). The FVM has enjoyed use in gas dynamics related to mechanical and aerospace engineering applications, but it has not seen widespread use in hydrodynamic modeling. The method applies local conservation of mass and momentum by integrating over a discrete volume and then solves for the boundary fluxes on each volume using an approximate Riemann solver, such as Roe's linearization. Advantages include its ability to capture shocks without introducing spurious oscillations, local and global mass conservation, utilization of the primitive form of the equations, and the ability to handle irregular meshes.

To our knowledge, phase behavior of the FVM, as applied to the system of SWE, has not been examined previously. In this work, we compare the phase properties of the FVM to other solution algorithms currently in use.

1.2 *Model backgrounds*

The system of SWE arise from the depth-averaged Navier-Stokes equations and require numerical techniques for solution over complex domains. Herein we will present a brief description of each of the algorithms that we examine in our study. Further details can be found in the references.

One of the first successful algorithms that did not produce spurious modes was the staggered finite difference (SFD) scheme (Leendertse 1967). This approach uses a staggered central difference stencil on a regular grid with the velocity evaluated at the grid interfaces and elevation centered in the middle of the grid block. This ensures perfect mass balance at local and global levels.

Before the introduction of the GWC equation, the primitive system of SWE were discretized using various finite elements, such as linear and higher order basis functions, quadratic and cubic isoparametric elements, and mixed interpolants. The discretization used for comparative purposes in this paper is the piecewise linear Galerkin finite element, which is

plagued by spurious oscillations. In future sections, this method will be noted as the Primitive FE (PFE) method.

Motivated by the generally poor results from the primitive forms of the SWE, another technique derived by Lynch and Gray (1979) was to modify the system of SWE before trying numerical solutions. They proposed to use their derived wave continuity equation instead of the primitive continuity in the shallow water model. This approach was further modified with the addition of a numerical G parameter to obtain the GWC equation (Kinnmark 1986), which is the form studied herein. The resulting GWC equation and the non-conservative momentum equation were then discretized using piecewise linear Galerkin finite elements. This method has proven to reduce the presence of spurious oscillations, yet maintains good conservation properties by using a numerical G parameter whose value varies for different physical simulations (Kolar et al. 1994). By increasing the magnitude of G, the GWC equation more closely resembles the primitive continuity form. The time-stepping scheme for the GWC utilized in this paper is centered at level k by setting the weighting coefficients (α_1, α_2, α_3) equal to 1/3.

Another finite element technique of the same generation is the selective lumping (SLFE) scheme of Kawahara et al. (1982). This is a two-step explicit scheme in which the mass matrix is selectively lumped by using a lumping parameter e with standard linear Galerkin elements. The parameter can take on values between 0.0 (consistent mass matrix) and 1.0 (fully lumped mass matrix), where in-between values "selectively" lump the mass matrix. In this paper we have generalized the lumping procedure by introducing a separate lumping parameter, f, for the mass matrix associated with the bottom friction term; this allows greater flexibility in the analyses.

Finally, the FVM has been applied to the primitive SWE by Chippada et al. (1998). This method discretizes the domain into finite volumes and solves the Riemann problem at each cell interface. Fluxes are computed at each face using an approximate Riemann solver, and then continuity principles are used to compute the resulting nodal values. Mass balance is preserved locally and globally. Low order schemes employ piecewise constant approximations within each volume, while higher order methods use piecewise linear (or higher) approximations. The nature of the discretization using high order methods does not lend itself to analytical techniques, so numerical approaches must be devised.

2 METHODS

2.1 *General equations and usage*

The linearized one-dimensional form of the equations governing shallow water flow are presented below.

The system of SWE includes the non-conservative momentum equation

$$u_t + \tau u + g\zeta_x = 0 \tag{1}$$

and the primitive continuity equation

$$\zeta_t + hu_x = 0 \tag{2}$$

where u is the depth-averaged velocity, ζ is the surface elevation, τ is the linear bottom friction factor, g is the acceleration of gravity, h is the bathymetric depth of water (assumed constant), and the subscripts indicate partial derivatives. The linearized GWC equation is expressed as

$$\zeta_{tt} + G\zeta_t + (G - \tau)hu_x - gh\zeta_{xx} = 0 \tag{3}$$

where G is the numerical coefficient that determines the balance between primitive (large values of G) and pure wave (small values of G) forms.

All of the solution algorithms presented in the previous section use equation (1) to solve for the velocity field at the next time level and equation (2) to solve for the surface elevation at the next time level, with the exception of the GWC approach that uses equation (3) to solve for surface elevation.

2.2 *Discrete models*

As the focus of this manuscript is the FVM, only its discrete algorithm is presented herein. For discrete equations of SFD, GWC, SLFE refer to Leendertse (1967), Luettich et al. (1991), and Kawahara et al. (1982), respectively. The primitive model follows as a limiting case of the GWC by letting $G \to \infty$.

The starting point for the derivation of the discrete, low-order FVM equations is the vector form of the primitive equations (1) and (2), i.e.,

$$\mathbf{c}_t + \mathbf{f}_x = \mathbf{r} \tag{4}$$

where

$$\mathbf{f} = \begin{pmatrix} hu \\ g\zeta \end{pmatrix}, \mathbf{c} = \begin{pmatrix} \zeta \\ u \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 0 \\ -\tau u \end{pmatrix},$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{c}} = \mathbf{A} = \begin{bmatrix} 0 & h \\ g & 0 \end{bmatrix} \tag{5}$$

Note that since u and ζ do not appear directly in $\mathbf{A}$ for the linear SWE, this is the same matrix as the Roe matrix in Leveque (1990). When working with the nonlinear equations, it would be necessary to find the Roe averaged variables using a linearization technique. The eigenvalues and eigenvectors of the matrix $\mathbf{A}$ are given by

$$\lambda_{1,2} = \mp\sqrt{gh}\,, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -a/h \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ a/h \end{pmatrix} \quad (6)$$

where $a = \sqrt{gh}$. The next step is to solve the Riemann problem at cell interfaces using

$$\mathbf{c}_R = \mathbf{c}_L + \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 \quad (7)$$

as the relationship between the left and right states at a point. By substituting the definitions of (5) and (6) into (7), it is possible to solve for the coefficients α_1 and α_2 in terms of the state variables, u and ζ. The numerical flux term is then approximated by

$$\mathbf{F} = \mathbf{f}_L + \sum_{i=1}^{2} min(\lambda_i, 0)\alpha_i \mathbf{v}_i \quad (8)$$

We now return to the system of conservation equations; the integral formulation of the system over a finite volume (element) is written as

$$\frac{\partial}{\partial t}\int_{\Omega_e} \mathbf{c}\,d\Omega_e + \int_{\Gamma_e} \mathbf{f}_n d\Gamma_e = \int_{\Omega_e} \mathbf{r}\,d\Omega_e \quad (9)$$

where Ω_e and Γ_e are the area and boundary of an element, and $\mathbf{f}_n$ is the normal flux across the boundary of the control volume. In one-dimension, the elements are line segments so the boundary integral reduces to point evaluations. Utilizing explicit time-stepping, the resulting discrete equations are

$$\frac{\mathbf{c}_e^{n+1} - \mathbf{c}_e^{n}}{\Delta t}\Delta x_e + \mathbf{F}_R^n - \mathbf{F}_L^n = \mathbf{r}^n \Delta x_e \quad (10)$$

where $\mathbf{c}_e$ is the average value of $\mathbf{c}$ in an element of width Δx_e and $\mathbf{F}$ and $\mathbf{r}$ are given by (8) and (5), respectively. Using these relations, explicit time-stepping, and choosing a piecewise constant representation of the variables in each control volume for low-order approximations, the discrete equations for the low-order FVM are expressed as

$$\zeta_i^{n+1} = \zeta_i^n - \frac{\Delta t}{2\Delta x}(h(u_{i+1}^n - u_{i-1}^n)$$

$$-\sqrt{gh}(\zeta_{i-1}^n - 2\zeta_i^n + \zeta_{i+1}^n)) \quad (11)$$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}(g(\zeta_{i+1}^n - \zeta_{i-1}^n)$$

$$- \sqrt{gh}(u_{i-1}^n - 2u_i^n + u_{i+1}^n)) - \tau u_i^n \Delta t \Delta x \quad (12)$$

In comparing the discrete equations (11) and (12)

with their continuum counterparts (1) and (2), we note that the low-order FVM is basically a central difference approximation that introduces a second order term into the discrete equations, viz, those terms preceded by $\sqrt{gh}$ in (11) and (12). Kinnmark (1986) and others have noted that this is a characteristic feature of the GWC (and other non-oscillatory) algorithm. In other words, the presence of a second order space derivative is correlated with the ability of an algorithm to successfully propagate or damp out the undesirable short wavelength oscillations. Subsequent analyses will test this hypotheses. For a more detailed derivation of the FVM applied to the nonlinear SWE, see Chippada et al. (1998).

2.3 Fourier analysis

2.3.1 Derivation
Fourier analysis is a useful tool to analyze the propagation behavior of both analytical and discrete solutions of homogeneous, linear difference or differential equations. Any solution that is piecewise continuous on a finite interval or is periodic on an infinite interval can be represented by a Fourier series expansion. If the solution (numerical or analytical) is a function that satisfies these minimal constraints, then it can be represented as a complex Fourier series

$$f(x, t) = \sum_{n=-\infty}^{\infty} A_n e^{i\alpha_n t} e^{i\beta_n x} \quad (13)$$

where, for the SWE, $f = u$ or ζ, $i = \sqrt{-1}$, α_n is the temporal frequency of the solution, $\beta_n = 2\pi/L_n$ is the wave number, and A_n is the Fourier coefficient for component n. The key to Fourier analysis is that the linearity of the differential equation allows one to examine a single Fourier component of the solution at a time. This type of analysis is frequently referred to as von Neumann analysis.

The starting point for Fourier analysis is the discretized system of equations. A single component of the Fourier series, represented as

$$f_j^k = f_0 e^{i\alpha_n k \Delta t} e^{i\beta_n j \Delta x} \quad (14)$$

where $f = u$ or ζ is substituted into the discrete algorithms for each independent variable. A set of simultaneous equations for the independent variables results. A nontrivial solution is sought for the homogeneous system, which requires the determinant of the coefficient matrix to be zero. Next define the discrete propagation factor

$$\lambda \equiv \frac{f_j^{k+1}}{f_j^k} = e^{i\alpha_n \Delta t} \quad (15)$$

and substitute (15) into the set of equations for the algorithm under study. A polynomial in the propagation factor λ results, which can then be solved as a function of the wave number.

The damping and phase properties of the numerical solution are determined by computing $|\lambda|$ and $\tan\theta \equiv \mathrm{Im}\lambda/\mathrm{Re}\lambda$, respectively. The discrete results are then compared against the analytical propagation factor, which has a magnitude of 1. In this paper, the damping error is reported as a per timestep comparison. Thus, damping error for this study is defined as

$$\text{Damping ratio per timestep} \equiv \frac{|\lambda|}{|\Lambda|}, \qquad (16)$$

where $|\Lambda| = 1$ for the analytical solution.

The phase error is reported as a per wavelength comparison. Thus, the phase error is computed after the analytical solution travels one wavelength, that is, it is computed after the solution travels N_n time steps where $N_n = L_n / \sqrt{gh}\Delta t$. Analytically, a wave experiences a phase change of 2π after N_n steps. Numerically, a wave propagates $N_n\theta$ in the same time where

θ is computed from the phase expression above. With this, the phase error is given by

$$\text{Phase error per wavelength} \equiv N_n\theta - 2\pi. \qquad (17)$$

For a difference algorithm to perfectly match the analytical solution of the model problem, the damping ratio should be 1.0 and the phase error should equal 0.0 for all components of the solution. Note that the phase error can be positive or negative, indicating phase lead or phase lag, respectively.

2.3.2 Application

The polynomial expressions for the propagation factor, λ, resulting from the above analysis steps are summarized in Table 1 for each of the algorithms under study. Note that the expressions for the SFD and the Primitive FE differ only in the constants S, T, W, and Z. Also, the expressions for the FVM and SLFE methods are similar in form.

Parameter values for the algorithms in Table 1 for a typical physical simulation are $h = 10$m, $g = 9.81$ m/s^2, $G = 0.001$ s^{-1}, $\tau = 0.0001$s^{-1}, $\Delta x = 1000$m, $\Delta t = 1$s,

Table 1. Polynomial expressions for the propagation factor, λ, from Fourier analysis

SFD	$[Tr - S\theta^2]\lambda^2 + [T(-2r + \tau\Delta t) - S(2\theta - 2\theta^2)]\lambda + [T(r - \tau\Delta t) - S(1 - \theta)^2] = 0$
PFE	$[Wr - Z\theta^2]\lambda^2 + [W(-2r + \tau\Delta t) - Z(2\theta - 2\theta^2)]\lambda + [W(r - \tau\Delta t) - Z(1 - \theta)^2] = 0$
SLFE	$\lambda^2 + [2M + \tau\Delta t\bar{F}]\lambda + M[M + \tau\Delta t\bar{F}] - \frac{1}{4}\mu^2 gh\left(E^2 - \frac{1}{2}\tau\Delta tEF\right) = 0$
GWC	$P_3\lambda^3 + P_2\lambda^2 + P_1\lambda + P_0 = 0$
Low-order FVM	$\lambda^2 + [-2s + \tau\Delta t]\lambda + \left[s(s - \tau\Delta t) - \frac{1}{4}hg(\mu B)^2\right] = 0$

$A = e^{i\beta\Delta x} + e^{-i\beta\Delta x}$, $B = e^{i\beta\Delta x} - e^{-i\beta\Delta x}$, $C = e^{i2\beta\Delta x} + e^{-i2\beta\Delta x}$, $D = e^{-i2\beta\Delta x} - e^{i2\beta\Delta x}$,

$E = \frac{1-e}{6}D - \frac{2+e}{3}B$, $F = \frac{1-f}{6}D - \frac{2+f}{3}B$, $\bar{E} = \frac{1-e}{6}A + \frac{2+e}{3}$, $\bar{F} = \frac{1-f}{6}A + \frac{2+f}{3}$,

$M = -\frac{\mu^2}{8}gh(C - 2) - \bar{E}$, $r = 1 + \tau\Delta t\theta$, $s = 1 + \frac{\mu}{2}a(A - 2)$, $S = 2i\sin\left(\frac{1}{2}\beta\Delta x\right)g\mu^2 h(e^{i\beta\Delta x} - 1)$,

$T = e^{(i\beta\Delta x)/2}$, $W = (A + 4)^2$, $Z = 9g(\mu B)^2 h$, $\mu = (\Delta t)/(\Delta x)$, $K_3 = \frac{\Delta x^2}{3\Delta t^3}$, $K_2 = \frac{\Delta x^2}{3\Delta t^2}$, $K_1 = \frac{\Delta x^2}{6\Delta t}$,

$P_0 = -K_3\left(2 + \frac{A}{2}\right) + (K_2(G + \tau) - K_1 G\tau)\left(1 + \frac{A}{4}\right) + \alpha_3 gh\left(1 - \frac{A}{2}\right)\left(\tau - \frac{2}{\Delta t}\right)$,

$P_1 = 3K_3\left(2 + \frac{A}{2}\right) - (K_2(G + \tau) + K_1 G\tau)\left(1 + \frac{A}{4}\right) + gh\left(1 - \frac{A}{2}\right)\left(\frac{2}{\Delta t}(\alpha_3 - \alpha_2) + \tau(\alpha_3 + \alpha_2)\right) - \frac{gh}{8}B^2(G - \tau)$,

$P_2 = -3K_3\left(2 + \frac{A}{2}\right) - (K_2(G + \tau) - K_1 G\tau)\left(1 + \frac{A}{4}\right) + gh\left(1 - \frac{A}{2}\right)\left(\frac{2}{\Delta t}(\alpha_2 - \alpha_1) + \tau(\alpha_2 + \alpha_1)\right) - \frac{gh}{8}B^2(G - \tau)$,

$P_3 = K_3\left(2 + \frac{A}{2}\right) + (K_2(G + \tau) + K_1 G\tau)\left(1 + \frac{A}{4}\right) + \alpha_1 gh\left(1 - \frac{A}{2}\right)\left(\tau + \frac{2}{\Delta t}\right)$

and $\theta = 0.5$ for Crank-Nicholson time-stepping. The resulting damping and phase behavior plots are presented in Figures 1 and 2, respectively.

2.4 Dispersion analysis

2.4.1 Derivation

This analysis tool utilizes the assumption that the solution of the differential equations can be separated and expressed as periodic in time and space. It varies from Fourier analysis in that the time variable remains continuous by using the harmonic form of the equations. The analysis results in a dispersion curve, which is a plot of the magnitude of the temporal frequency versus wave number. Platzman (1981) was the first to apply the analysis technique to the shallow water equations; he showed that a monotonic curve indicates a numerical solution free from spurious $2\Delta x$ oscillations. A folded curve indicates aliasing of wave components with one wave corresponding to the long physical wave while the other corresponds to short-wavelength noise in the solution. It has been demonstrated elsewhere that for linear elements, a folded curve is a necessary and sufficient condition for the appearance of spurious modes in the numerical solution of the SWE, but not for quadratic or other elements. (Kinnmark 1986).

The harmonic form of each equation is attained by substituting the harmonic components expressed as

$$\zeta = \bar{\zeta}e^{i\omega t} \text{ and } u = \bar{u}e^{i\omega t} \tag{18}$$

into the differential equations. Here $\bar{\zeta}$ is the spatial harmonic of the elevation, $\bar{u}$ is the spatial harmonic of the velocity, $i = \sqrt{-1}$, ω is the temporal frequency of the solution, and t is time. The next step is to discretize each equation in space according to the solution algorithm. Then harmonic spatial solutions expressed as

$$\bar{\zeta} = \zeta_0 e^{ij\sigma\Delta x} \text{ and } \bar{u} = u_0 e^{ij\sigma\Delta x} \tag{19}$$

are substituted into the discrete equations. Here also $i = \sqrt{-1}$, j is the spatial node index, σ is the spatial frequency of the solution (the wave number), Δx is the discrete spacing of nodes, ζ_0 is the magnitude of the elevation solution and u_0 is the magnitude of the velocity solution.

From the substitution of (19) into the discrete equations, a new system results

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \zeta_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{20}$$

where the form of the matrix coefficients a, b, c, and d will vary depending upon the spatial discretization. A non-trivial solution will exist only when the determinant is zero. Expansion results in a polynomial expression in ω as a function of the wave number, which is then solved for its roots. The magnitude of the roots vs. wave number are plotted as a dispersion curve. For more background information on dispersion analysis see (Platzman 1981, Foreman 1983).

2.4.2 Application

The dispersion relations derived using the above technique are summarized in Table 2 for each of the algorithms under study. The analytic dispersion relation was derived by substituting harmonics into the continuum equation.

Table 2. Polynomial expressions for temporal frequency, ω, from dispersion analysis

Analytical	$\omega^2 + [-i\tau]\omega + [\sigma^2 gh] = 0$
SFD	$\omega^2 + [-i\tau]\omega + \left[\dfrac{hg}{\Delta x^2}(A-2)\right] = 0$
PFE	$\omega^2 + [-i\tau]\omega + \left[9\dfrac{gh}{\Delta x^2}\left(\dfrac{B}{3A+2B}\right)^2\right] = 0$
SLFE	$\omega^2 + [-2iJ - i\tau\bar{F}]\omega + \left[\dfrac{gh}{4\Delta x^2}(E^2 - EF) - J^2 - \tau J\bar{F}\right] = 0$
GWC	$\left[\dfrac{-i}{6}(4+A)\right]\omega^3 + \left[\dfrac{-i}{6}(G+\tau)(4+A)\right]\omega^2 + \left[\dfrac{i}{6}G\tau(4+A) + \dfrac{ihg}{\Delta x^2}(2-A)\right]\omega$ $+ \left[\dfrac{hg\tau}{\Delta x^2}(2-A) + \dfrac{hg}{4\Delta x^2}(\tau - G)B^2\right] = 0$
Low-order FVM	$\omega^2 + \left[-i\tau + \dfrac{ia(A-2)}{\Delta x}\right]\omega + \left[-\left(\dfrac{a(A-2)}{2\Delta x}\right)^2 + \dfrac{a\tau(A-2)}{2\Delta x} + \dfrac{1}{4}gh\left(\dfrac{B}{\Delta x}\right)^2\right] = 0$
	$J = \dfrac{gh\Delta t}{8\Delta x^2}(2-C) - \dfrac{1}{\Delta t}(\bar{E}+1)$

Note that the GWC solution algorithm results in a cubic polynomial with three roots. For a 1-D problem, two of these are physical roots, while the third is a numerical artifact from the derivative formulation of the GWC equation (Kinnmark 1986). Additionally, note that the time discretization method of the SLFE does not treat the mass matrix consistently between terms from the same time derivative. Thus, it is not possible to keep the time variable continuous and still maintain the character of the selective lumping algorithm. In order to cast the equation into the time-continuous form needed for dispersion analysis, we were forced to split the time derivative operator into two parts. One part was left as a continuous differential operator and the other was discretized using the selective lumping technique of Kawahara et al. (1982). Because of this approximation, a Δt artifact remained in the harmonic equations, which was parameterized with the other variables. All other parameter values are the same as in the Fourier analysis. The resulting dispersion curves are presented in Figure 3.

3 DISCUSSION

3.1 *Fourier analysis*

Recall that the desired phase behavior of a difference algorithm is that the damping ratio should be 1.0, while the phase error is 0.0 for all components of the solution. This indicates that those components that are in-phase with the analytical solution are not damped out of the numerical solution. If the numerical algorithm is unable to propagate short waves, then they must be damped in order to avoid spurious modes. Overdamping occurs when the algorithm also damps the predominant long physical waves.

Figures 1 and 2 show the Fourier phase behavior. The GWC and SFD are able to propagate all waves with only a small amount phase lag or lead for short waves; there are no zero phase velocities (Figure 2). Also, the GWC and SFD have only a small amount of damping for short waves (Figure 1). Thus, the algorithms are able to produce accurate, noise-free solutions. On the other hand, primitive finite element models show up to a 2π phase error (Figure 2) for short waves less than $10\Delta x$ i.e, they do not propagate short waves well. (Note that a 2π phase error corresponds to a zero phase velocity.) Furthermore, PFE models do not damp these short waves (Figure 1). Hence, spurious modes appear in the solution because energy that accumulates at this end of the spectrum (through round-off error, nonlinear interactions, or initial conditions) is neither propagated nor damped.

Selective lumping (with both lumping parameters set to 0.9) also does not propagate short waves (Figure 2), but it does damp out the noise, as seen in Figure 1. However, the SLFE overdamps in that some predom-

inant longer physical waves are also attenuated, as can be seen by damping ratios less than one for waves greater than $10\Delta x$. Such overdamping is not desirable. Note that the amount of damping can be controlled through the selection of the lumping parameter. However the phase behavior (Figure 2) for the SLFE algorithm does not change for different parameter values even though the damping is removed for higher parameter values. Thus, to remove all of the damping by allowing the lumping parameters to equal 1.0 would introduce aliasing of waves, which is shown here in the Fourier analysis, as well as in the dispersion analysis below. Finally, we note that the low-order FVM, the subject of this manuscript, behaves similarly to the SLFE with lumping parameters set to 0.9, viz, short waves are poorly propagated (Figure 2), but they are also damped (Figure 1). However, in contrast to SLFE, the damping is more controlled, i.e., less damping of the longer waves, as can be seen in Figure 1 where the FVM curve is to the left of the SLFE curve.

3.2 *Dispersion analysis*

Recall that a folded dispersion curve indicates spurious modes in the numerical solution. The analytical dispersion relation, plotted as the dashed line in Figure 3, is a straight line with slope $\sqrt{gh}$. Ideally, the best numerical solution would most closely approximate that relation. Note that the smallest resolvable wavelengths are of length $2\Delta x$, which, for the parameter values herein, corresponds to a σ-value of $(2\pi)/(2\Delta x) = 0.00314$, such that the small wavelengths are represented in the rightmost portion of Figure 3.

As noted in the introductory comments, the PFE algorithm results in a folded dispersion curve, which indicates that aliasing will occur. Recall that the SLFE dispersion relation is somewhat of an approximation, as discussed above. However, even with the manipulations, the dispersion analysis produced meaningful results in that they corroborated findings from the Fourier analysis above. The SLFE algorithm is monotonic for the unlumped (consistent) mass matrix, but as the lumping parameters increases toward 1.0 (fully lumped), the dispersion curve folds. Based on numerical simulations, Kawahara et al. (1982) recommended that the lumping parameter be chosen between 0.8 and 0.9. This range is within the limits where the dispersion curve is monotonic, but it does not approximate the analytical dispersion relation well (this also confirms the poor damping behavior seen in the Fourier analysis).

The GWC discretization results in a monotonic curve for the G parameter value examined in this study. However, if G becomes too large, then the equation approaches the primitive form, the dispersion curve folds over, and aliasing occurs. This is why Kolar et al. (1994) recommended an upper bound on

G of $10\tau_{max}$, where τ is the bottom friction factor.

The FVM and SFD discretization methods result in monotonic dispersion curves; interestingly these curves are nearly identical. Upon examination of the equations in Table 2, it is not obvious why this might be the case. In any event, at the smaller wavelengths near $2\Delta x$, the curves deviate from the analytical relation, but they do not fold. Thus, some error is introduced, but the analysis indicates solutions should be noise-free (simulation results bear this out). Note that when the scale is expanded to include wavelengths up to Δx, both dispersion relations (FVM and SFD) fold over. Thus, it becomes apparent that the FVM and SFD discretization techniques effectively shift the noise below the resolvable grid scale of the numerical approximation.

4 CONCLUSIONS

The finite volume method, as applied to the shallow water equations, is an attractive algorithm for model development. The low order method exhibits good phase behavior, a monotonic dispersion relation, and is inherently mass conservative. Based on simulation results with the Chippada et al. (1998) code, we expect that higher order methods, wherein dependent variables are interpolated with linear or higher order polynomials, would exhibit even better phase behavior. However, since the solution algorithm, in particular the slope limiter step, evolves with the solution itself, it does not lend itself to an analytical study of its phase behavior. Consequently, we are exploring techniques to carry out the study numerically.

Additionally, due to the space constraints of this manuscript, we were unable to present sensitivity of the phase behavior of the various algorithms to the choice of numerical parameters. In particular, it would be instructive to determine the correlation between parameter values that optimize the phase propagation behavior of the algorithm (as represented by Fourier and dispersion analyses), and simulation results. Furthermore, analysis of the discrete phase behavior for the low-order FVM algorithm may suggest optimum values for the Courant number (based on linear wave celerity, the Courant number is given as $C_r = \sqrt{gh}\Delta t/\Delta x$), while analysis of the high-order FVM would allow us to evaluate the impact of slope limiters on the phase behavior.

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Table 1. Polynomial expressions for the propagation factor, λ, from Fourier analysis

SFD	$[Tr - S\theta^2]\lambda^2 + [T(-2r + \tau\Delta t) - S(2\theta - 2\theta^2)]\lambda + [T(r - \tau\Delta t) - S(1 - \theta)^2] = 0$
PFE	$[Wr - Z\theta^2]\lambda^2 + [W(-2r + \tau\Delta t) - Z(2\theta - 2\theta^2)]\lambda + [W(r - \tau\Delta t) - Z(1 - \theta)^2] = 0$
SLFE	$\lambda^2 + [2M + \tau\Delta t\bar{F}]\lambda + M[M + \tau\Delta t\bar{F}] - \frac{1}{4}\mu^2 gh\left(E^2 - \frac{1}{2}\tau\Delta t EF\right) = 0$
GWC	$P_3\lambda^3 + P_2\lambda^2 + P_1\lambda + P_0 = 0$
Low-order FVM	$\lambda^2 + [-2s + \tau\Delta t]\lambda + \left[s(s - \tau\Delta t) - \frac{1}{4}hg(\mu B)^2\right] = 0$
	$A = e^{i\beta\Delta x} + e^{-i\beta\Delta x},\ B = e^{i\beta\Delta x} - e^{-i\beta\Delta x},\ C = e^{i2\beta\Delta x} + e^{-i2\beta\Delta x},\ D = e^{-i2\beta\Delta x} - e^{i2\beta\Delta x},$ $E = \frac{1-e}{6}D - \frac{2+e}{3}B,\ F = \frac{1-f}{6}D - \frac{2+f}{3}B,\ \bar{E} = \frac{1-e}{6}A + \frac{2+e}{3},\ \bar{F} = \frac{1-f}{6}A + \frac{2+f}{3},$ $M = -\frac{\mu^2}{8}gh(C - 2) - \bar{E},\ r = 1 + \tau\Delta t\theta,\ s = 1 + \frac{\mu}{2}a(A - 2),\ S = 2i\sin\left(\frac{1}{2}\beta\Delta x\right)g\mu^2 h(e^{i\beta\Delta x} - 1),$ $T = e^{(i\beta\Delta x)/2},\ W = (A + 4)^2,\ Z = 9g(\mu B)^2 h,\ \mu = (\Delta t)/(\Delta x),\ K_3 = \frac{\Delta x^2}{3\Delta t^3},\ K_2 = \frac{\Delta x^2}{3\Delta t^2},\ K_1 = \frac{\Delta x^2}{6\Delta t},$ $P_0 = -K_3\left(2 + \frac{A}{2}\right) + (K_2(G + \tau) - K_1 G\tau)\left(1 + \frac{A}{4}\right) + \alpha_3 gh\left(1 - \frac{A}{2}\right)\left(\tau - \frac{2}{\Delta t}\right),$ $P_1 = 3K_3\left(2 + \frac{A}{2}\right) - (K_2(G + \tau) + K_1 G\tau)\left(1 + \frac{A}{4}\right) + gh\left(1 - \frac{A}{2}\right)\left(\frac{2}{\Delta t}(\alpha_3 - \alpha_2) + \tau(\alpha_3 + \alpha_2)\right) - \frac{gh}{8}B^2(G - \tau),$ $P_2 = -3K_3\left(2 + \frac{A}{2}\right) - (K_2(G + \tau) - K_1 G\tau)\left(1 + \frac{A}{4}\right) + gh\left(1 - \frac{A}{2}\right)\left(\frac{2}{\Delta t}(\alpha_2 - \alpha_1) + \tau(\alpha_2 + \alpha_1)\right) - \frac{gh}{8}B^2(G - \tau),$ $P_3 = K_3\left(2 + \frac{A}{2}\right) + (K_2(G + \tau) + K_1 G\tau)\left(1 + \frac{A}{4}\right) + \alpha_1 gh\left(1 - \frac{A}{2}\right)\left(\tau + \frac{2}{\Delta t}\right)$

Table 2. Polynomial expressions for temporal frequency, ω, from dispersion analysis

Analytical	$\omega^2 + [-i\tau]\omega + [\sigma^2 gh] = 0$
SFD	$\omega^2 + [-i\tau]\omega + \left[\frac{hg}{\Delta x^2}(A - 2)\right] = 0$
PFE	$\omega^2 + [-i\tau]\omega + \left[9\frac{gh}{\Delta x^2}\left(\frac{B}{3A + 2B}\right)^2\right] = 0$
SLFE	$\omega^2 + [-2iJ - i\tau\bar{F}]\omega + \left[\frac{gh}{4\Delta x^2}(E^2 - EF) - J^2 - \tau J\bar{F}\right] = 0$
GWC	$\left[\frac{-i}{6}(4 + A)\right]\omega^3 + \left[\frac{-i}{6}(G + \tau)(4 + A)\right]\omega^2 + \left[\frac{i}{6}G\tau(4 + A) + \frac{ihg}{\Delta x^2}(2 - A)\right]\omega$ $\quad + \left[\frac{hg\tau}{\Delta x^2}(2 - A) + \frac{hg}{4\Delta x^2}(\tau - G)B^2\right] = 0$
	$J = \frac{gh\Delta t}{8\Delta x^2}(2 - C) - \frac{1}{\Delta t}(\bar{E} + 1)$

Table 2. Polynomial expressions for temporal frequency, ω, from dispersion analysis

Low-order FVM	$\omega^2 + \left[-i\tau + \dfrac{ia(A-2)}{\Delta x} \right]\omega + \left[-\left(\dfrac{a(A-2)}{2\Delta x}\right)^2 + \dfrac{a\tau(A-2)}{2\Delta x} + \dfrac{1}{4}gh\left(\dfrac{B}{\Delta x}\right)^2 \right] = 0$
	$J = \dfrac{gh\Delta t}{8\Delta x^2}(2-C) - \dfrac{1}{\Delta t}(\bar{E}+1)$