Appeared in: *Computational Methods in Water Resources XIII, Vol. 2,* Bentley et al., eds. 899-906, Balkema, 2000.

A look back at 20 years of GWC-based shallow water models

R.L. Kolar
School of Civil Engineering and Environmental Science, University of Oklahoma, Norman, OK 73019

J.J. Westerink
Dept. of Civil Engineering and Geological Sciences, University of Notre Dame, Notre Dame, IN 46556

ABSTRACT: Wave continuity equation models were introduced in 1979 by Lynch and Gray as a means to suppress spurious oscillations that appeared in finite element models based on the primitive equations. Since then, the basic algorithm has been studied and tested extensively; with subsequent modifications and extensions, it has proven to be a computationally efficient, reliable algorithm for predicting the hydrodynamic behavior of coastal and oceanic waters. Modifications over the last 20 years can be broken into two broad, but inter-related classes: 1) changes to the numerical algorithm, e.g., implicit time-marching or higher order elements; 2) incorporating more of the underlying physics into the mathematical formulation, e.g., baroclinic forcing or wetting and drying of near-shore elements. This manuscript summarizes the algorithm's evolution and highlights some its strengths as well as its shortcomings. It closes with some observations about the future of shallow water algorithms based on the generalized wave continuity (GWC) equation.

1 INTRODUCTION

The year 1999 marked 20 years since the publication of the seminal work of Lynch & Gray (1979) on the wave continuity equation, which represented the first non-oscillatory finite element scheme for the shallow water equations. Given all the developments over the intervening 20 years, it seems appropriate to review its origins, modifications, and applications from our current perspective. Moreover, a comprehensive understanding of the wave continuity algorithm will allow for more meaningful decisions about future directions in shallow water modeling.

This paper follows the development of the algorithm in more or less chronological order. First, we look at early finite element simulators in general, emphasizing their inadequacies. Such shortcomings prompted a long search by Gray, and later Gray & Lynch, for a better finite element algorithm. The "early years," from 1977 to 1984, focused on analyzing, developing, and evaluating a non-oscillatory, non-dissipative solution strategy. Next came the "middle years," from 1985 to 1995, which consisted of further analysis and development, as well as more realistic applications (concomitant with advances in computer hardware and software). From 1995 on, we have witnessed an significant increase in the variety and number of applications, ranging from three-dimensional larval transport studies to hurricane storm surges over intricate levied systems. Hand-in-hand with the applications are model development issues, viz, applications point out model shortcomings, which require more development and analysis. Finally, we close with an assessment of the state of wave continuity models and speculate on the future of shallow water modeling, taking into account the rapidly-changing nature of high performance computing.

2 TERMINOLOGY

Shallow water models solve the depth-integrated forms of the conservation of mass and momentum equations under the assumption of a hydrostatic pressure distribution. Using operator notation, where L represents the primitive form of the continuity equation and $\mathbf{M}^C$ the conservative form of the momentum equation, the wave continuity equation, as originally presented in Lynch & Gray (1979), is given as

$$W \equiv \frac{\partial L}{\partial t} + \tau L - \nabla \bullet \mathbf{M}^c = 0 \qquad (1)$$

where τ is the bottom friction factor. Equation (1) is called a wave continuity equation because in simplified form, it represents a damped, second order wave equation for surface elevation, ζ, viz,

$$\frac{\partial^2 \zeta}{\partial t^2} + \tau \frac{\partial \zeta}{\partial t} - gh\nabla^2 \zeta = \text{rhs forcing} \qquad (2)$$

The generalized form of the wave continuity equation is formed by replacing τ in (1) with a numerical parameter, G, i.e.,

$$W^G \equiv \frac{\partial L}{\partial t} + GL - \nabla \bullet \mathbf{M}^c = 0 \qquad (3)$$

The G-parameter determines the balance in the wave continuity equation between the primitive and pure wave form. Its value has no physical meaning and is only used to control the numerical properties of the solution. The higher the magnitude of G, the more the generalized wave continuity equation approaches the primitive continuity equation. In the limit, $W^G \rightarrow L$ as $G \rightarrow \infty$. From hereon, (3) is referred to as the generalized wave continuity (GWC) equation, unless $G = \tau$ in which case it is referred to as the wave continuity equation.

The discrete model is typically found by discretizing the GWC and momentum equation in space using a standard Galerkin finite element approximation with linear triangular elements. Implicit time discretization of the GWC equation uses a three-time-level approximation centered at k, while the momentum equation (in either conservative or nonconservative form) uses a lumped two-time-level approximation centered at k + 1/2. Nonlinear terms are evaluated explicitly, and exact quadrature rules are employed. The resulting equations are reported elsewhere (e.g., Luettich et al. 1991) so the lengthy results are not given here. A split step, sequential solution procedure is adopted wherein the GWC equation is solved for elevation, and the momentum equation is solved for the velocity field.

3 PRE-WAVE CONTINUITY MODELS (< 1977)

Prior to 1970, numerical shallow water models fell almost exclusively in the domain of finite differences (FD), and of these, Leendertse's (1967) staggered grid model became the standard. However, with the recognition that the finite element (FE) method can be viewed as a general procedure for solving partial differential equations, researchers began to explore finite element solutions to the shallow water equations. A primary reason these early finite element efforts were undertaken is that the developers recognized the value of using unstructured grids to more accurately map irregular coastal geometries; finite element grids are also easier to refine. Grotkop (1973) is often credited with the first attempt to develop a workable finite element-based simulator. He used six node space-time finite elements to solve the primitive equations and applied the algorithm to the North Sea using 69 spatial nodes. No field plots are given, but he does claim to show good agreement with measured observations. However, later analyses showed this solution to be excessively dissipative (Gray 1982).

Over the next few years, a number of other finite element models were reported in the literature. Taylor & Davis (1975) looked at both quadratic and cubic isoparametric elements to discretize the space domain, and they examined three different time marching algorithms: fourth order Adams-Moulton multi-step predictor-corrector, Crank-Nicholson, and finite elements in time using both linear Lagrange and Hermitian cubic interpolants. Wang & Connor (1975) used linear triangles for the space discretization and various time marching algorithms, including a split step finite difference scheme. Partridge & Brebbia (1976) discretized the space domain with quadratic triangles and looked at both Runge-Kutta and implicit Euler time-marching algorithms. Three characteristics are common to all of these early efforts: 1) use of Galerkin's method for the spatial discretization; 2) tremendous concern over the type of time marching algorithm; 3) appearance of short wave ($2\Delta x$) noise in the solution, particularly for the more "real-world" problems with 2D flow structure and non-constant bathymetry. It appears that the focus on time marching algorithms was primarily in response to the third problem, viz, a search for an algorithm that would propagate or dissipate the short wavelength components of the solution without affecting the longer wavelength components. None of these early modelers came across such an algorithm, so they resorted to suppressing the short waves using a variety of techniques: artificially high viscous or friction coefficients; overly dissipative time marching algorithms; smoothing (averaging) of the solution in a post-processing step. A secondary reason for the time stepping studies was the fact that computer memory at that time was on the order of kilobytes, so large sparse matrices could not be stored or inverted.

In the mid-1970's, Gray also recognized the inherent flexibility of the finite element method for simulating geophysical flows and joined the search for a "workable" finite element shallow water model. His first attempts are reported in a 1977 USGS manuscript (Gray 1977). Of utmost concern was that the resulting algorithm be both accurate and cost-competitive with FD codes. Gray's initial work was strongly influenced by the staggered finite difference modelers. However, algorithmic "tricks," which seemed almost trivial in a FD framework, did not transfer easily to the FE framework. Consider some of the hurdles. Limited computer memory often dictated the numerical algorithm. To minimize matrix storage and inversion, FD modelers resorted to alternating direction implicit methods, which are based on tridiagonal algorithms. Gray attempted to mimic this approach by examining three different split scheme algorithms: Leendertse's two-step scheme, Abbott's two-step scheme, and a four-step scheme developed by Gray. All were abandoned because of the difficulty in imposing normal flux boundary conditions. In particular, with FD models on rectangular grids, the normal flux is always parallel to one of the axes, but with irregular FE meshes, the flux need not be parallel, and most often is not. Gray could not develop a split scheme algorithm, wherein *U* and *V* are solved for sequentially, without introducing some bias in the solution or without distorting the flow field near land boundaries. It is interesting to note that one of the options that he considered was to treat flux conditions as natural in the continuity equation instead of essential for the momentum equation, an approach that was also used by Partridge & Brebbia (1976), but it was not theoretically justified until 1996 (Kolar et al. 1996).

Guided by Taylor & Davis' (1975) work with the Navier-Stokes equations, Gray did look at mixed interpolation (*H* linear and *U,V* quadratic) in an attempt to rid simulations of spurious modes. Results were much improved, but he still observed non-physical sea surface elevations. Next, Gray tried mixed interpolation (linear/quadratic isoparametric elements) in conjunction with a fully implicit treatment of the governing equations, using a Picard-like iteration and relaxation to resolve the nonlinear terms. It was with this algorithm that he developed a physically-consistent treatment of the normal flow boundary conditions by rotating the momentum equation to normal-tangential coordinates on the boundary. The slow execution time (1-10 sec per time step on an IBM 360) and "large" memory requirements (150K for 119 nodes) led him to conclude that an implicit procedure could not be cost competitive with FD models.

Consequently, Gray settled on an explicit model based on quadratic isoparametric elements with a leap-frog time marching algorithm. Mass matrices were lumped "naturally" through the use of nodal quadrature. One additional FE difficulty that Gray did address with this algorithm is a consistent treatment of flux boundary terms when there is a discontinuity in the shoreline angle at a node point. He concluded that the problem should be addressed by finding conditions on the rotation angles such that the integral of the normal flux over the boundary segment adjacent to the nodal point should be zero. Applications indicated the algorithm had promise, but oscillations were still observed in surface elevations.

4 THE EARLY YEARS (1977-1984)

Gray renewed his efforts at finding a non-oscillatory FE solution when, in 1977, he and Lynch published a study on 10 different time stepping algorithms for the shallow water equations. The study was restricted to one-dimensional linearized equations on a uniform grid, which were discretized in space using linear Galerkin FE. Fourier analysis, first used by Leendertse to study FD algorithms, was used to study the amplitude and phase portraits of 10 different time marching algorithms: Crank-Nicholson, linear FE in time, leap-frog, split-step, second order Adams-Bashforth, second order Adams-Bashforth partially-corrected, Lax-Wendroff, 3-level semi-implicit, 2 level semi-implicit, and wave equation (implicit and explicit). Lynch & Gray recognized that in order for an algorithm to be accurate and oscillation free, then it should mimic the behavior of the analytical solution, viz, the magnitude of the propagation factor and the phase ratio should be close to 1.0 for all wavelengths. In other words, both short waves and long waves should propagate through the system with the amount of dissipation controlled by friction and viscous terms, not by the time marching procedure. On the contrary, if short waves do not propagate, then there must be some numerical damping of just the short waves in order to avoid oscillations. None of the first six algorithms mentioned above, all of which are based on the primitive equations, met these conditions, i.e., they could not propagate or damp the short waves properly.

The last four schemes mentioned above all share a common trait: they are derived by manipulating the governing equation through differentiation and substitutions so that the algorithm solves a mathematically equivalent problem, yet one that displays different numerical properties. In general, all four show better phase propagation characteristics than the first six, but the Lax-Wendroff is too heavily damped. The semi-implicit schemes have less damping, but the overall phase and amplitude behavior is not as good as the wave equation. In fact, the wave equation is the only one that has no artificial damping nor 2π phase lag for the shortest resolvable wave (for a range of Courant numbers). A common characteristic of the first three is the appearance of a second order space derivative in either the algorithm to solve for surface elevation or velocity.

Thus the 1977 Gray & Lynch work represented a fundamental breakthrough in the search for non-oscillatory FE solutions. It is not clear from the article whether the Lax-Wendroff scheme or the three-level semi-implicit, which was developed by Kwizak & Robert (1971) for atmospheric modeling, served as the inspiration for the wave equation. Regardless, it became clear to them that the key to proper phase behavior *in the context of Galerkin's FE method* was to manipulate the governing equations. Gray & Lynch also realized that the semi-implicit and wave approaches would lend themselves to a sequential solution procedure and explicit treatment of the nonlinear terms, two factors that were of great concern, given the computer resources of that day.

Gray & Lynch published a set of companion articles in 1979 that further explored the two most promising algorithms from their 1977 study: the three-level semi-implicit model and the wave continuity model. In the first, Gray & Lynch (1979) compared a 2D primitive equation leapfrog model with the semi-implicit one. Spatial derivatives in both models were approximated using Galerkin FE on linear triangles or quadratic quadrilaterals. Both time-stepping schemes allowed for a sequential solution procedure with the nonlinear terms evaluated explicitly. Nodal quadrature was used to diagonalize the mass matrices. Fourier analysis shows that the primitive equation model neither damps nor propagates $2\Delta x$ waves, whereas the semi-implicit model damps, but does not propagate the short waves. Numerical tests confirmed analysis results, namely, numerical noise was a real problem with the primitive equation model for all but the simplest domains, whereas it was much less pronounced in the semi-implicit model. However, noise was still apparent in some of the two-dimensional experiments. From this study, they concluded that a non-oscillatory Galerkin FE algorithm must damp and propagate $2\Delta x$ waves, yet maintain high accuracy for the predominant physical waves.

Lynch & Gray (1979) then examined the response characteristics of the nonlinear 2D wave continuity equation, whose linear 1D form met the desired characteristics of a non-oscillatory algorithm (Gray & Lynch 1977). As with the three-level semi-implicit model of Kwizak & Robert (1971), the key is to manipulate the governing equations so that a second derivative of surface elevation appears in the continuity equation (refer to the Introduction for the equations). Galerkin FE was used to discretize the space domain (linear triangles and quadratic quadrilaterals), and a three-time-level FD algorithm in the time domain. Nonlinear terms were evaluated explicitly, and the elevation/velocity fields were solved sequentially. In two of the algorithms, nodal quadrature was used to diagonalize the mass matrices (lumped explicit and lumped semi-implicit); the other algorithm was fully explicit with consistent matrices. In this original wave continuity formulation, the bottom friction appeared as a time-dependent coefficient in the mass matrix. Lynch & Gray made the matrices stationary by solving for intermediate variables, and then post-processing these to back out the elevation and velocity field. Numerical experiments confirmed analysis results in that it produced accurate, noise-free solutions in all test cases.

Following publication of the wave continuity algorithm, a flurry of articles appeared that furthered the analyses of Lynch & Gray. They often included a comparison to the only other promising primitive FE algorithm at that time - mixed finite elements that attempted to mimic the successful staggered FD codes (e.g., Walters & Cheng 1979). All these significant works looked at the response characteristics in the frequency domain via Fourier transforms or modal (dispersion) analysis.

Platzman's pioneering work appeared in 1981 and was the first to introduce the concept of aliasing as applied to the shallow water models of that day. Instead of just looking at the steady-state modal behavior ($\omega = 0$ where ω is the temporal frequency), Platzman analyzed time-dependent modes by looking at the full spectrum via a dispersion curve (wave number-frequency relation graph). He found that primitive equation models with equal-order interpolants have a folded dispersion relation so that two wave numbers are associated with each frequency - one is the long physical wave and the other is the short non-physical wave (near $2\Delta x$) that can be introduced through nonlinear interactions, among other sources. On the other hand, wave continuity models have a monotonic dispersion relation (one wave number per frequency) that mimics the analytical dispersion relation. Platzman also generalized the wave continuity approach under a general category he termed "derivative models," which, in this case, can be derived by representing velocity by Stokes/Helmholtz potentials ($u = -h\partial\phi/\partial x$ in 1D). Thus, the wave continuity "trick" belongs to the same class of methods in classical fluid mechanics where the continuity equation for incompressible flow becomes Laplace's equation for velocity potential, the latter being much easier to work with numerically.

Foreman (1983) expanded the dispersion analyses of the wave continuity equation to include a general family of two-step (three level) time marching algorithms. Using dispersion and asymptotic analyses of consistent and lumped formulations, he concluded that accuracy, stability, and efficiency is optimized with the explicit, lumped wave continuity model.

Others (Williams & Zienkiewicz 1981, Walters & Carey 1983, Walters 1983, Walters & Carey 1984) examined the dispersion relation of equal-order FE, mixed FE, staggered FD and FE, or the wave continuity algorithm. Significant conclusions include the following: All primitive models using equal-order interpolation are doomed to spurious oscillation modes; Staggered grids (FE or FD) can produce noise-free solutions by effectively shifting noise to wavelength Δx, which is below the resolvable grid scale, yet these staggered methods are difficult to implement with unstructured meshes; Mixed FE with elevation interpolated with piecewise constants and velocity with linear polynomials works well in 1D, but can lead to the continuity equation being underconstrained in 2D (in 2D, the number of nodal points does not maintain a constant relationship with the number of element centers, as it does in 1D); Mixed FE with elevation interpolated with linear polynomials and velocity with quadratics produces noise free elevations, but Walters (1983) notes that noise still appears in the velocity field. Thus, Walters (1983) concludes that because wave continuity models propagate short waves, energy won't accumulate in this high frequency end of the spectrum so no subgrid dissipation mechanism is needed. For other FE algorithms, this is not the case so artificial or numerical damping must be introduced.

Gray's 1982 work is the most complete FE inter-model comparison at that time, and it further highlighted the last point made in the paragraph above. In that study, he examined seven algorithms in addition to the wave continuity and noted primary shortcomings. His main finding was that only the wave continuity algorithm produces noise free solutions.

Gray did check mass conservation in the 1982 study for steady flow through a constricted channel; it was satisfactory so the issue was not pursued. Walters & Carey (1984) appear to be first to raise the question of mass imbalance with the wave continuity algorithm, where, in the last statement of their conclusions, they note that although the wave continuity produces a monotonic dispersion relation, "the method can lead to continuity problems since only the time rate of change of the continuity equation is required to vanish." This observation would prove to be prophetic, as will be seen in the next decade.

While it may appear to be a footnote with regard to the topic of this manuscript, finite element grids have actually had a tremendous impact on algorithm development. During the 1977-1984 time period, automatic meshing algorithms were limited, so many grids were still constructed by hand. Consequently, grids were coarse (a large grid may have contained 100 nodes), bathymetry depths were interpolated graphically, and coast lines were not finely resolved. As a consequence, algorithms were not "pushed" hard in that they were simulating slowly evolving flow features on coarse regions of relatively open water.

5 THE MIDDLE YEARS (1985 -1995)

Kinnmark's (1986) monograph, in terms of scope and lasting contributions, represents one of the most complete studies of the wave continuity algorithm to date. Much of the work was devoted to refined stability and accuracy studies of explicit and implicit formulations on both uniform and nonuniform grids. These studies confirmed the superior wave propagation characteristics of the wave continuity formulation. Two items have had particularly lasting significance. First, he introduced the generalized wave continuity equation (see (3)) where the weight associated with the primitive continuity equation, G, is distinct from the bottom friction factor, τ. The flexibility to select G distinct from τ has had far-reaching consequences in terms of minimizing errors and improving model performance. An obvious advantage readily noted by Kinnmark is that it also makes the GWC mass matrix time invariant. Second, using operator notation, he examined the equivalence between the primitive equations and other formulations, including the wave continuity equation. The operators allow for an elegant and very general method to describe a wide class of equations. He concluded that the generalized wave continuity equation is analytically equivalent to the primitive continuity equation provided $L(0) = 0$, that is the initial conditions satisfy the primitive continuity equation. This condition arises because differentiating a differential equation can open up the solution space to include a wider class of solutions (e.g., compare the general solutions of $u' = 0$ with $u'' = 0$). If the condition $L(0) = 0$ is not met, as indeed may be the situation in numerical applications, then he concluded the following: 1) if the conservative form of the momentum equation is used, then non-zero disturbances will be damped out whenever $G > 0$ (this is always true in applications); 2) if the non-conservative form of the momentum equation is used, then non-zero initial disturbances will be damped out whenever $G - \nabla \bullet \mathbf{v} > 0$. The latter condition cannot be guaranteed *a priori*, so G should be chosen as large as possible without introducing spurious oscillations.

Because the GWC algorithm propagates short waves, no artificial damping is needed. Because in most hydrodynamic simulations of estuarine and coastal waters, internal viscous stresses are much less in magnitude than bottom stresses, no viscous dissipation terms were included in early GWC models. However, it became apparent in two applications in the late 1980's that the observed flow field could not be simulated without including viscous stresses: Lynch's effort to simulate gyres in the Mediterranean and Kolar and Gray's effort to simulate circulation patterns in trout broodstock ponds. A problem arose when it was proposed to model viscous dissipation using a simple eddy viscosity term where losses are proportional to the Laplacian of the velocity field as this would introduce a second order derivative (even after application of Green's Theorem) in the GWC equation, which could not be interpolated using the standard C^0 Lagrange basis functions. Lynch (1988) solved the problem by introducing a momentum dissipation term (proportional to the Laplacian of the velocity field) that he evaluated external to the GWC equation. Kolar and Gray (1990) solved the problem by using the continuity equation to replace the second order space derivative with a space and time derivative. Both produced acceptable results and both have been adopted in production codes.

Gross mass balance errors (as measured by a direct integration of the primitive continuity equation) in transient simulations were discovered independently by Westerink and Kolar in 1992. Westerink found that in simulations of flow around inlets and barrier islands in the Gulf of St. Lawrence, mass was not being conserved. Kolar noticed the mass balance errors when simulating circulation patterns in the aforementioned trout broodstock ponds, a much different application of the shallow water equations than tidal hydrodynamics in that the length scale is an order of magnitude less. The similarity between the two is that the nonlinear terms, particularly the advective terms, were significant in both applications. It was also found that mass balance errors serve as a surrogate variable for other problems, such as amplitude and phasing errors. Extensive numerical experiments and analysis indicated that the source of error was indeed the nonlinear terms. The key to minimizing error, which was reported in Kolar et al. (1994) was to use a value of $G > \tau$ so that the weight given to the primitive continuity equation (see (3)) is nontrivial, but not so large as to cause the fundamental character of the equation to be primitive. It is interesting to note that this confirms Walters and Carey's (1984) hunch that the vanishing of $\partial L / \partial t$ alone, which is the case when $G = \tau$, is not sufficient to ensure global conservation. Moreover, this finding supports Kinnmark's analysis that G must be large in order to maintain an equivalency between the primitive and GWC equation.

Two other algorithmic changes helped improve mass conservation, which were reported in Kolar et al. (1994): 1) reformulate the advective terms in the GWC so that they are in non-conservative form (consistent with the momentum equation); 2) use mass conservative boundary conditions (as first reported in Lynch 1985) where normal fluxes are treated as natural conditions in the continuity equation. We note that the dispersion relation once again played a crucial role in GWC analysis, viz, it was used in Kolar et al. (1994) to identify the maximum value that could be specified for G before the dispersion relation became aliased (folded), thus ensuring noise-free solutions with minimal mass balance error.

Increased computer capacity allowed additional physics to be incorporated into the simulation. In particular, by splitting field calculations into external (vertically-averaged equations of motion) and internal (velocity profile over the vertical) modes, pseudo-3D simulations could be conducted. Two techniques for resolving the velocity profile emerged: velocity-based and stress-based. The former has been widely used in FD, spectral, and FE models, while the latter was developed by Luettich & Westerink (1991) and Luettich et al. (1994) and is used in the GWC-based ADCIRC model. An advantage of the stress-based algorithm is that fewer node points are required near the bottom boundary because shear stress varies approximately linearly near the boundary layer. In contrast velocity varies logarithmically for a no-slip bottom boundary condition (a slip condition at the bottom helps mitigate this resolution problem).

Boundary conditions received additional attention during this period as well. Traditionally, water surface elevation is specified on open boundaries and treated as an essential (Dirichlet) condition for the continuity equation, while zero normal flux is specified on land boundaries and treated as a natural (Neumann) condition for the momentum equation. But, with coupled equations for the elevation and velocity field, it is not clear if this is the proper way to interpret boundary conditions. For example, a zero flux condition can be interpreted as a Dirichlet condition on the momentum equation (normal component) or a Neumann condition on the continuity equation or both. Drolet and Gray (1988) used characteristic theory to examine the *number* of conditions that must be applied, but did not specify how to impose them. Lynch (1985) indicated that global mass conservation can only be realized in primitive equation or wave equation models by using the continuity equation to compute normal fluxes on boundaries where surface elevation is specified, and then use the computed flux in velocity calculations. Westerink et al. (1994) examined the relation between boundary conditions and spurious modes, which indirectly impacts mass balance.

Advents in mesh generating software automated the otherwise tedious process of grid development. The increased resolution of the finer, shallower coastal features increased the significance of the nonlinear terms in those regions, particularly the advective terms. In order to maintain stable simulations with GWC-based models, a severe Courant restriction had to be imposed and parameter values had to be chosen carefully.

6 THE PRESENT (1996-2000)

Advances in computer hardware have continued to drive many development efforts in shallow water models. GWC-based codes have incorporated more and more physics, including the following significant capabilities: three-dimensional baroclinic simulations, sediment transport, wetting/drying of near-shore areas, and barrier boundaries that include levees (with or without culverts) and floodwalls. The modifications are driven by the increased use of shallow water models for effective coastal planning and storm warnings by federal, state, and local governments, as well as private consulting groups.

Calculation of the vertical velocity was re-examined during this era. The problem is inherently ill-posed in that the velocity is governed by a first order equation, yet two boundary conditions exist (free surface and bottom boundary). Most early algorithms used the derivative approach, in which a second derivative operator was applied to the governing equation to convert the equation to second order so that the problem was well-posed. Muccino (1997) showed that the method can lead to gross mass balance errors (analogous to GWC mass balance errors). To correct the problem, she proposed using the adjoint method to calculate the vertical velocity

(Muccino 1997); it has shown to produce mass conservative results.

Recent algorithmic work has focused on one of three themes: time stepping, parallel computing, and mass balance. After the flurry of analysis and experiments during the early and middle years, it seems that GWC-based models settled on a semi-implicit, split level time marching algorithm. In particular, the GWC equation is solved for elevation using a three-time-level scheme centered at k, while the momentum equation is solved for the velocity field using a two-level scheme centered at $k + 1/2$. Nonlinear terms are evaluated explicitly. Kolar et al. (1998) and Dresback & Kolar (1999) proposed a two-stage, predictor-corrector algorithm to treat the nonlinear terms implicitly. Numerous experiments in 1D and 2D have demonstrated a significant increase in the maximum stable time step.

Although many of the GWC-based models had been optimized for vector and scalar machines, little effort had been devoted to parallelizing the algorithm. Chippada et al. (1996) used domain decomposition techniques to parallelize the code for clusters of workstations or massively parallel computers. To minimize inter-processor communication, they used a Hilbert space-filling curve to divide the domain; results show a nearly linear scale-up. Parr (1999) further refined the parallel algorithm by introducing an alternative domain decomposition technique (metis algorithm) and by rewriting large sections of code. Benchmarking results on a wide variety of platforms, ranging from massively parallel supercomputers to clusters of workstations, have approached theoretical limits (actually superlinear speed-up for some problems, depending on cache size).

Current releases of mesh generating software allow users to create highly irregular meshes of complicated regions in a relatively automated fashion - a far cry from the hand generated meshes just 20 years ago. Highly resolved bathymetric and coastline maps are also easily incorporated into the process. This, coupled with reliable algorithms and advanced post-processors (and even whole integrated modeling packages), has allowed users to dramatically increase the number and type of applications to which GWC-based models are applied. Tidal circulation, the original motivation for the GWC algorithm, is no longer the primary application; rather, applications now cover the gamut, ranging from larval transport to sediment transport to Naval fleet operations to coastal dredging operations.

Simulations of ever larger domains with increased resolution has proven to be a real challenge for GWC-based models, particularly from a mass balance point of view. Highly nonlinear flows, such as shallow, converging sections around barrier islands or flood waves propagating onto dry land, typically signifies regions where local mass imbalances may occur. To complement previous algorithmic changes to improve mass balance (see the previous section), researches took a renewed look at boundary conditions. In particular, Kolar et al. (1996) used generalized functions to study the impact of boundary conditions on mass conservation. They concluded that specified normal fluxes should be treated as Neumann conditions in the continuity equation in order to realize global mass conservation. Lynch and Holboke's (1997) rigorous study reinforced this conclusion. They extended the analysis to 3D models and stipulated that Neumann conditions should also be enforced strongly on the velocity solution in order to realize higher accuracy and maintain a reversibility between Dirichlet and Neumann conditions. An interesting outcome of their analyses, for both time continuous and discrete problems, is that the GWC algorithm is mass conservative provided either the initial conditions satisfy the primitive continuity equation or $G > 0$ so that perturbations (due to initial conditions or roundoff error) are damped. This is identical to Kinnmark's (1986) finding discussed earlier. Most recently, Westerink (1999) exploited this property by implementing a spatially-variable G, which allows the user to use a larger value of G (i.e., more primitive) in nonlinear regions where mass imbalance is a problem.

7 THE FUTURE?

Has the GWC algorithm outlived its usefulness? We think not. With split-step time marching, it is a very fast, efficient algorithm. It lends itself well to parallel and vector optimization. With the algorithmic changes over the years, it can accurately simulate a wide range of problems. It's a tribute to the groundwork laid by Gray & Lynch that the GWC algorithm, which was originally developed for simulating long water waves at a relatively coarse scale, is capable of handling the demands placed on it by today's sophisticated applications. Its current shortcomings stem from two problems: 1) instabilities and mass imbalance for highly nonlinear flow; 2) supercritical flow (e.g., dam breaks). Thus, we anticipate the next-generation of GWC models will have an option for using an alternative algorithm (which will have its own set of limitations), such as the finite volume method, on those portions of the domain not amenable to the GWC algorithm. Such an approach would lend itself well to parallel processing/domain decomposition techniques (the future of high performance computing) wherein different algorithms could be applied to different subdomains.

ACKNOWLEDGMENTS

Funding is provided in part through the National Science Foundation, contract #ACI-9623592.

REFERENCES

Chippada, S., C.N. Dawson, M.F. Wheeler, & M.L. Martinez, "Parallel Computing for Finite Element Models of Surface Water Flow," *CMWR XI, Volume 2: Computational Methods in Surface Flow and Transport Systems*, Aldama et al., eds., 63-70, 1996.

Dresback, K.M., & R.L. Kolar, "Implicit Time-Marching Algorithm for Shallow Water Models Based on the Generalized Wave Continuity Equation," *International Journal for Numerical Methods in Fluids*, in review, Oct. 1999.

Drolet, J., & W.G. Gray, "On the Well Posedness of Some Wave Formulations of the Shallow Water Equations," *Advances in Water Resources*, 11, 84-91, 1988.

Foreman, M.G.G., "An Analysis of the Wave Equation Model for Finite Element Tidal Computations," *Journal of Computational Physics*, 52, 290-312, 1983.

Gray, W.G., "Progress Report: Estuary Modeling by the Finite Element Method," USGS, Water Resources Division, Reston, Virginia, 76 pp., 1977.

Gray, W.G., "Some Inadequacies of Finite Element Models as Simulators for Two-Dimensional Circulation," *Advances in Water Resources*, 5, 171-177, Sept., 1982.

Gray, W.G., & D.R. Lynch, "Time-Stepping Schemes for Finite Element Tidal Model Computations," *Advances in Water Resources*, 1(2), 83-95, 1977.

Gray, W.G., & D.R. Lynch, "On the Control of Noise in Finite Element Tidal Computations: A Semi-Implicit Approach," *Computers and Fluids*, 7, 47-67, 1979.

Grotkop, G., "Finite Element Analysis of Long-Period Water Waves," *Computer Methods in Applied Mechanics and Engineering*, 2, 147, 1973.

Kinnmark, I., *The Shallow Water Wave Equations: Formulation, Analysis, and Application*, Lecture Notes in Engineering, 15, Springer-Verlag, Berlin, 187 pp., 1986.

Kolar, R.L., J.P. Looper, J.J. Westerink, & W.G. Gray, "An Improved Time Marching Algorithm for GWC Shallow Water Models," *CMWR XII: Vol. 2 - Computational Methods in Surface and Ground Water Transport*, Burganos et al. eds., Computational Mechanics, Southampton, 379-386, 1998.

Kolar, R.L., & W.G. Gray, "Shallow Water Modeling in Small Water Bodies," *Computational Methods in Surface Hydrology*, Gambolati et al., eds., Computational Mechanics Publications/Springer-Verlag, Southampton/Berlin, 149-155, 1990.

Kolar, R.L., J.J. Westerink, M.E. Cantekin, & C.A. Blain, "Aspects of Nonlinear Simulations Using Shallow Water Models Based on the Wave Continuity Equation," *Computers and Fluids*, 23(3), 523-538, 1994.

Kolar, R.L., W.G. Gray, & J.J. Westerink, "Boundary Conditions in Shallow Water Models - An Alternative Implementation for Finite Element Codes," *International Journal of Numerical Methods in Fluids*, 22, 603-618, 1996.

Kwizak, M., & A.J. Robert, "A Semi-Implicit Scheme for Grid Point Atmospheric Models of the Primitive Equations," *Monthly Weather Review*, 99, 32, 1971.

Leendertse, J.J., "Aspects of a Computational Model for Long-Period Water Wave Propagation", *Memorandum RM-5294-PR*, Rand Corporation, 1967.

Luettich, R.A., & J.J. Westerink, "A Solution for the Vertical Variation of Stress, Rather than Velocity, in a Three-Dimensional Circulation Model," *International Journal for Numerical Methods in Fluids*, 12, 911-928, 1991.

Luettich, R.A., S. Hu, & J.J. Westerink, "Development of the Direct Stress Solution Technique for Three Dimensional Hydrodynamic Models Using Finite Elements," *International Journal for Numerical Methods in Fluids*, 19, 295-319, 1994.

Luettich, R.A., Jr., J.J. Westerink & N.W. Scheffner, *ADCIRC: An Advanced Three-Dimensional Circulation Model for Shelves, Coasts, and Estuaries*, Dept. of the Army, U.S. Army Corps of Engineers, Washington, D. C., August, 1991.

Lynch, D.R., "Mass Balance in Shallow Water Simulations," *Communications in Applied Numerical Methods*, 1, 153-159, 1985.

Lynch, D.R. & M.J. Holboke, "Normal Flow Boundary Conditions in 3D Circulation Models," *International Journal for Numerical Methods in Fluids*, 25, 1185-1205, 1997.

Lynch, D.R., & W.G. Gray, "A Wave Equation Model for Finite Element Tidal Computations," *Computers and Fluids*, 7(3), 207-228, 1979.

Lynch, D.R., F.E. Werner, A. Cantos-Figuerola, & G. Parrilla, "Finite Element Modeling of Reduced-Gravity Flow in the Alboran Sea: Sensitivity Studies," *Proceedings: Workshop on the Gibraltar Experiment*, Madrid, Spain, 1988.

Muccino, J.C., W.G. Gray, & M.G.G. Foreman, "Calculation of the Vertical Velocity in Three-Dimensional, Shallow Water, Finite Element Models," *International Journal for Numerical Methods in Fluids*, 22, 779-802, 1997.

Parr, V., Personal Communication, 1999.

Partridge, P.W., & C.A. Brebbia, "Quadratic Finite Elements in Shallow Water Problems," *Journal Hydraulics Division, ASCE*, 102, HY9, 1299-1313, 1976.

Platzman, G., "Some Response Characteristics of Finite Element Tidal Models," *J Computational Physics*, 40(1), 36-63, 1981.

Taylor, C., & J.M. Davis, "Tidal and Long-Wave Propagation: A Finite Element Approach," *Computers and Fluids*, 3(2-3), 125-148, 1975.

Walters, R.A., "Numerically Induced Oscillations in Finite Element Approximations to the Shallow Water Equations," *International Journal of Numerical Methods in Fluids*, 3(6), 591-604, 1983.

Walters, R.A., & G.F. Carey, "Analysis of Spurious Oscillation Modes for the Shallow Water and Navier-Stokes Equations," *Computers and Fluids*, 11(1), 51-68, 1983.

Walters, R.A., & G.F. Carey, "Numerical Noise in Ocean and Estuarine Models," *Advances in Water Resources*, 7, 15-20, March, 1984.

Walters, R.A., & R.T. Cheng, "Accuracy of an Estuarine Hydrodynamic Model Using Smooth Elements," *Water Resources Research*, 16(1), 187-195, 1980.

Walters, R.A., & R.T. Cheng, "A Two-Dimensional Hydrodynamic Model of a Tidal Estuary," *Advances in Water Resources*, 2, 177-184, 1979.

Wang, J.D., & J.J. Connor, "Mathematical Modeling of Near Coastal Circulation," MIT Parsons Laboratory Report #200, Cambridge, MA, 1975.

Westerink, J.J., R.A. Luettich, Jr., J.K. Wu, and R.L. Kolar, "The Influence of Normal Flow Boundary Conditions on Spurious Modes in Finite Element Solutions to the Shallow Water Equation," *International Journal for Numerical Methods in Fluids*, 18, 1201-1060, 1994.

Williams, R.T., & O.C. Zienkiewicz, "Improved Finite Element Forms for the Shallow-Water Wave Equations," *Interna-*

tional Journal for Numerical Methods in Fluids, 1, 81-97, 1981.